格林函数在困难中应用

2022.3.10.

Field-theoretical methods in the many-body problem

§1. The Schrödinger, Heisenberg, and interaction pictures

① Schrödinger picture: 波函数 $\Psi_S(t)$

$$i\hbar \frac{\partial \Psi_S(t)}{\partial t} = H(t)\Psi_S(t)$$

The dynamical variable are take to be time independent.

(The Hamiltonian may contain an explicit time dependence)

If H is independent of time

$$\Psi_S(t) = e^{-iH(t-t_0)}\Psi_S(t=t_0)$$

② Heisenberg picture:

The wave function Ψ_H is time independent.

The time dependence of the problem is transferred to the operators.

可观测量随时间演化，态不随时间演化.

$$\frac{dA^{(H)}}{dt} = \frac{1}{i\hbar}\left[A^{(H)}, H\right]$$

↳ 满足海森伯运动方程

How to derive?

Θ_S 不随时间变化

$$\Theta_H(t) = e^{iH(t-t_0)/\hbar}\,\Theta_S\,e^{-iH(t-t_0)/\hbar}$$

$$\Psi_H(t) = e^{iH(t-t_0)/\hbar}\,\Psi_S(t) \Leftarrow \text{选择 } \Psi_H(t_0) = \Psi_S(t_0)$$

$$i\hbar\frac{\partial\Theta_H(t)}{dt} = [\Theta(t), H]$$

③. 相互作用表象

$$H = H_0 + H'$$

在相互作用绘景中，态和算符均随时间变化

$$\Psi_I(t) = e^{iH_0(t-t_0)/\hbar}\,\Psi_S(t)$$

$$\Theta_I(t) = e^{iH_0(t-t_0)/\hbar}\,\Theta_S(t)\,e^{-iH_0(t-t_0)/\hbar}$$

若 $H'=0$，相互作用绘景与 Heisenberg 绘景是一样的

$\Psi_I(t)$ 仅在 H' 下演化

$$i\hbar\frac{\partial\Psi_I(t)}{\partial t} = H_I'(t)\,\Psi_I(t)$$

$$H_I'(t) = e^{iH_0(t-t_0)/\hbar}\,H_S'\,e^{-iH_0(t-t_0)/\hbar}$$

证明：（现场演示）

多体问题 (many-body problem)

H' 较强, N 较多

⟹ 求解多体波函数非常困难 (除个例外)

⟹ 一种普遍的方法是避免求多体波函数, 而找寻与实验可观测量直接相关的"格林函数"

(目前强关联数值计算普遍努力的方向还是求解多体波函数, 但受制于计算能力)

格林函数可提供的信息:

① spin, charge densities

② momentum distribution of electrons

③ excitation spectrum of the system.

④ free-energy of the system.

⑤ electric conductivity, magnetic/spin susceptibility, and many other linear-response quantity.

⑥ ~~dynamics (~~ far-away from equilibrium

It is so useful such that every student in condensed matter physics should be familiar with it.

The greatest advantage: the approximation scheme is well developed, insight into many-body problem.

考虑 spin-$\frac{1}{2}$ 相互作用的费米子.（因课时原因这里不再讲述玻色子，但考试、作业时不限于费米子）

路径积分

计算 Metallic magnetism, Josephson 效应等

(外加 trivial 例: plasma)

2016.8.31

Propagator

量子力学

$$G(x(t), x(t_0)) = \boxed{\int_{x(t_0)}^{x(t)} \mathcal{D}x}\ e^{\frac{i}{\hbar}\int_{t_0}^{t} dt\,(p\dot{x} - H)}$$

$$\boxed{\langle x(t) | e^{-i\hat{H}t} | x(t_0) \rangle}$$

精神实质

相干态完备基

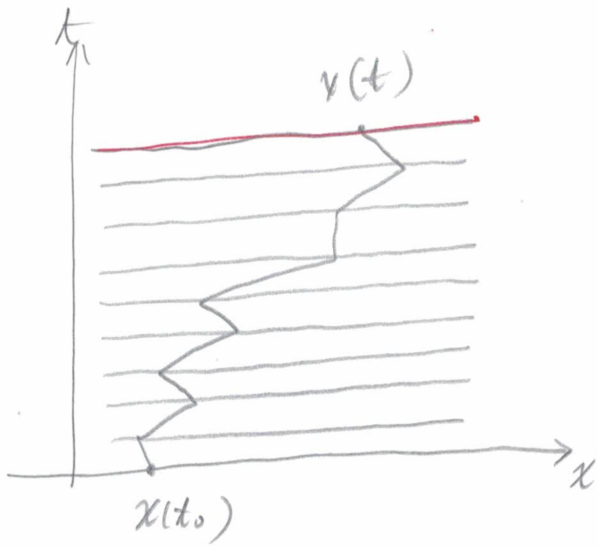
$$\mathrm{tr} A = \sum_n \langle n | A | n \rangle$$

插入相干态完备基

$$= \sum_n \int \frac{d^2\alpha}{\pi} \langle n | \alpha \rangle \langle \alpha | A | n \rangle$$

$$= \sum_n \int \frac{d^2\alpha}{\pi} \langle \alpha | A | n \rangle \langle n | \alpha \rangle$$

$$= \int \frac{d^2\alpha}{\pi} \langle \alpha | A | \alpha \rangle$$



对所有可能路径进行积分

什么

n个 Gauss 积分

One-dimensional $\int_{-\infty}^{+\infty} dx\, e^{-\frac{a}{2}x^2} = \sqrt{\frac{2\pi}{a}}$, $Re(a) > 0$

$$\int_{-\infty}^{+\infty} dx\, e^{-\frac{a}{2}x^2} x^2 = \sqrt{\frac{2\pi}{a^3}}$$

$$a \to a + \epsilon$$

$$\int_{-\infty}^{+\infty} dx\, e^{-\frac{a+\epsilon}{2}x^2} = \sqrt{\frac{\pi}{a+\epsilon}}$$

$$\int_{-\infty}^{+\infty} dx\, e^{-\frac{a}{2}x^2} \cdot e^{-\frac{\epsilon}{2}x^2}$$

$$= \int_{-\infty}^{+\infty} dx\, e^{-\frac{a}{2}x^2}\left(1 - \frac{\epsilon}{2}x^2 + \cdots\right)$$

$$= \frac{\sqrt{\pi}}{\sqrt{a+\epsilon}} = \sqrt{\frac{\pi}{a}} \cdot \left(1 + \frac{\epsilon}{a}\right)^{-\frac{1}{2}} = \sqrt{\frac{\pi}{a}}\left(1 - \frac{1}{2}\frac{\epsilon}{a} + \cdots\right)$$

因此

$$\int_{-\infty}^{+\infty} dx\, e^{-\frac{a}{2}x^2} - \frac{\epsilon}{2}\int_{-\infty}^{+\infty} dx\, e^{-\frac{a}{2}x^2} x^2 + \cdots$$

$$= \sqrt{\frac{\pi}{a}} - \frac{\epsilon}{2}\sqrt{\frac{\pi}{a^3}} + \cdots$$

因此 $\int_{-\infty}^{+\infty} dx\, e^{-\frac{a}{2}x^2} x^2 = \sqrt{\frac{\pi}{a^3}}$

(或者对 a 求导数)

有线性项的情况

$$\int_{-\infty}^{+\infty} dx\, e^{-\frac{a}{2}x^2 + bx} = \int_{-\infty}^{+\infty} dx\, e^{-\frac{a}{2}(x+\frac{b}{a})^2 + b(x+\frac{b}{a})}$$

$$= \int_{-\infty}^{+\infty} dx\, e^{-\frac{a}{2}x^2 - bx - \frac{b^2}{2a} + bx + \frac{b^2}{a}}$$

$$= \int_{-\infty}^{+\infty} dx\, e^{-\frac{a}{2}x^2 + \frac{b^2}{2a}} = \sqrt{\frac{2\pi}{a}}\, e^{\frac{b^2}{2a}}$$

变量为复数的情况

$$\int d(\bar{z}, z)\, e^{-\bar{z}\omega z}$$

$z = x + iy$, $\bar{z} = x - iy$

$\bar{z}\omega z = (x - iy)(x + iy)\omega$

$= (x^2 + y^2)\omega$

$$dz\,d\bar{z} = (dx+idy)(dx-idy) \quad \times \quad d(z,\bar{z}) \equiv \int_{-\infty}^{+\infty} dx\,dy$$

$$\cancel{= dx\,dx}$$

$$dx\,dy \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial \bar{z}}{\partial x} & \frac{\partial \bar{z}}{\partial y} \end{vmatrix} = dx\,dy \begin{vmatrix} 1 & i \\ 1 & -i \end{vmatrix} = dx\,dy \, \| -2i \| = 2\,dx\,dy \quad \times$$

$$\int d(z,\bar{z})\, e^{-\bar{z}wz} = \frac{\pi}{w}, \quad Re(w) > 0$$

$$\int d(\bar{z}, z)\; e^{-\bar{z}wz + \bar{u}z + v\bar{z}}$$

$$= \int dx\,dy\; e^{-(x^2+y^2)w + \bar{u}(x+iy) + v(x-iy)}$$

$$= \int dx\,dy\; e^{-(x^2+y^2)w + (\bar{u}+v)x + (i\bar{u}-iv)y}$$

$$= \int_{-\infty}^{+\infty} dx\; e^{-wx^2 + (\bar{u}+v)x} \cdot \int_{-\infty}^{+\infty} dy\; e^{-wy^2 + (i\bar{u}-iv)y}$$

$$= \sqrt{\frac{2\pi}{2w}}\; e^{\frac{(\bar{u}+v)^2}{2\cdot 2w}} \sqrt{\frac{2\pi}{2w}} \cdot e^{\frac{[i(\bar{u}-v)]^2}{2\cdot 2w}}$$

$$= \frac{\pi}{w}\; e^{\frac{\bar{u}^2 + 2\bar{u}v + v^2 - \bar{u}^2 + 2\bar{u}v + v^2}{4w}}$$

$$= \frac{\pi}{w}\; e^{\frac{4\bar{u}v}{4w}} = \frac{\pi}{w}\; e^{\frac{\bar{u}v}{w}} \qquad Re(w) > 0$$

$\boxed{\text{More than one dimension}}$

高维情况下的推广

实积分

$\vec{v}$ 为 N 维实向量，$\vec{A}$ 为 N×N 矩阵

$$\int d\vec{v}\, e^{-\frac{1}{2}\vec{v}^T\vec{A}\vec{v}}$$

$\vec{A} = O^T D O$，D 为对角阵

$\vec{A}$ 为实对称阵，正定

O 为正交变换

$\boxed{\det(O)=1}$

$$= \int d\vec{v}\, e^{-\frac{1}{2}\vec{v}^T O^T D O \vec{v}}$$

$$\vec{v} \to O\vec{v} \equiv \vec{v}'$$

$$= \int d\vec{v}'\, |O|^{-1} e^{-\frac{1}{2}\vec{v}'^T D \vec{v}'}$$

$$= \int d\vec{v}'\, e^{-\frac{1}{2}\vec{v}'^T D \vec{v}'} = \left(\sqrt{2\pi}\right)^N \left(\det\vec{A}\right)^{-\frac{1}{2}}$$

有线性项的情况

$$\int d\vec{v}\, e^{-\frac{1}{2}\vec{v}^T\vec{A}\vec{v} + \vec{j}^T\cdot\vec{v}}$$

$$\vec{v} \to \vec{v} + \vec{A}^{-1}\vec{j}$$

$$= \int d\vec{v}\, e^{-\frac{1}{2}(\vec{v}^T + \vec{j}^T(\vec{A}^{-1})^T)\vec{A}(\vec{v}+\vec{A}^{-1}\vec{j}) + \vec{j}^T(\vec{v}+\vec{A}^{-1}\vec{j})}$$

$$= \int d\vec{v}\, e^{-\frac{1}{2}(\vec{v}^T + \vec{j}^T(\vec{A}^{-1})^T)(\vec{A}\vec{v}+\vec{j}) + \vec{j}^T\vec{v} + \vec{j}^T\vec{A}^{-1}\vec{j}}$$

$$= \int d\vec{v}\, e^{-\frac{1}{2}(\vec{v}^T\vec{A}\vec{v} + \vec{v}^T\vec{j} + \vec{j}^T A\vec{v} + \vec{j}^T\vec{A}^{-1}\vec{j}) + \vec{j}^T\vec{v} + \vec{j}^T\vec{A}^{-1}\vec{j}}$$

而 $\vec{v}^T\vec{j} = \vec{j}^T\vec{v} = (\vec{v}^T\vec{j})^T$ 数

$$= \int d\vec{v}\, e^{-\frac{1}{2}\vec{v}^T \vec{A}\vec{v} + \frac{1}{2}\vec{j}^T \vec{A}^{-1}\vec{j}}$$

$$= (2\pi)^{\frac{N}{2}} \left(\det \vec{A}\right)^{-\frac{1}{2}} e^{\frac{1}{2}\vec{j}^T \vec{A}^{-1}\vec{j}}$$

Complex case <u>复变量情况</u>

$$\int d(\vec{v}^+, \vec{v})\, e^{-\vec{v}^+ \vec{A}\vec{v}} = \pi^N \det A^{-1}$$

$$\int d(\vec{v}^+, \vec{v})\, e^{-\vec{v}^+ \vec{A}\vec{v} + \vec{\omega}^+ \cdot \vec{v} + \vec{v}^+ \vec{\omega}'} = \pi^N \det \vec{A}^{-1} e^{\omega^+ A^{-1} \omega'}$$

$\boxed{\text{泛函积分}}$ $\mathcal{D}v(x) = \prod_{i=1}^{N} \mathcal{D}v_i$

$$\int \mathcal{D}v(x) \exp\left[-\frac{1}{2}\int dx\,dx'\, v(x) A(x,x') v(x') + \int dx\, j(x) v(x)\right]$$

$$= (2\pi)^{\frac{N}{2}} \det A^{-\frac{1}{2}}\, e^{\frac{1}{2}\int j(x) A^{-1}(x,x') j(x')\, dx\, dx'} = S$$

$\frac{1}{S}\int \mathcal{D}v(\tilde{x})\, v(x) v(x') \exp\left[-\frac{1}{2}\int d\tilde{x}\, d\tilde{x}'\, v(\tilde{x}) A(\tilde{x}, \tilde{x}') v(\tilde{x}') + \int d\tilde{x}\, j(\tilde{x}) v(\tilde{x})\right]$

本身这个泛函是发散的，但 $\langle v(x) v(x')\rangle$ 这样的项并不发散。

$$\langle \cdots \rangle \equiv (2\pi)^{-\frac{N}{2}} \det A^{\frac{1}{2}} \int \mathcal{D}v(x) \exp\left[-\frac{1}{2}\int dx\,dx'\, v(x) A(x,x') v(x')\right]$$

$\times (\cdots)$

$$\langle v(x) v(x')\rangle = \frac{\delta S}{\delta j(x)\,\delta j(x')} \cdot (2\pi)^{-\frac{N}{2}} \det A^{\frac{1}{2}}$$

$$= \frac{\delta}{\delta j(x)\,\delta j(x')} e^{\frac{1}{2}\int j(x'') A^{-1}(x'', x''') j(x''')\, dx''\, dx'''}$$

$$= \frac{\delta}{\delta j(x)} \cdot e^{\frac{1}{2}\int j(x'') A^{-1}(x'', x''') j(x''') dx'' dx'''} \left[\frac{1}{2}\int \delta(x'-x'') A^{-1}(x'', x''') \times j(x''') dx'' dx'''\right] + \frac{\delta}{\delta j(x)} e^{\frac{1}{2}\int j(x'') A^{-1}(x'', x''') j(x''') dx'' dx'''}$$

$$\times \left[\frac{1}{2}\int j(x'') A^{-1}(x'', x''') \delta(x'-x''') dx'' dx'''\right]$$

注意最后 j 要取0。

$$\Rightarrow A^{-1}(x, x')$$

2节课

Construction of the many body path integral.

<u>相干态</u> a 算符的本征态

Boson: $|\phi\rangle = \exp\left(\sum_i \phi_i a_i^{+}\right)|0\rangle$, $\boxed{i\text{为第}i\text{个粒子}}$

$\phi = \{\phi_i\}$ 代表一系列复数

$$a_i|\phi\rangle = a_i \exp\left(\sum_j \phi_j a_j^{+}\right)|0\rangle$$

$$= \left[a_i, \exp\left(\sum_j \phi_j a_j^{+}\right)\right]|0\rangle$$

$$= \left[a_i, \sum_n \frac{1}{n!}\left(\sum_j \phi_j a_j^{+}\right)^n\right]|0\rangle$$

$$\left(\sum_j \phi_j a_j^{+}\right)\left(\sum_j \phi_j a_j^{+}\right)^{n-1}$$

$$\left[a_i, \left(\sum_j \phi_j a_j^{+}\right)^n\right] = \left[a_i, \left(\sum_j \phi_j a_j^{+}\right)\right]\left(\sum_j \phi_j a_j^{+}\right)^{n-1} + \left(\sum_j \phi_j a_j^{+}\right)\left[a_i, \left(\sum_j \phi_j a_j^{+}\right)^{n-1}\right]$$

$$\| $$

$$n\phi_i \left(\sum_j \phi_j a_j^{+}\right)^{n-1}$$

$$\left[a_i, \sum_j \phi_j a_j^{+}\right] = \phi_i$$

$$\left[a_i, \left(\sum_j \phi_j a_j^{+}\right)^2\right] = \left(\sum_j \phi_j a_j^{+}\right)\left[a_i, \sum_j \phi_j a_j^{+}\right] + \left[a_i, \sum_j \phi_j a_j^{+}\right]\left(\sum_j \phi_j a_j^{+}\right)$$

$$= 2\phi_i \left(\sum_j \phi_j a_j^{+}\right)$$

因此

$$a_i|\phi\rangle = \sum_n \frac{n}{n!}\phi_i \left(\sum_j \phi_j a_j^\dagger\right)^{n-1}|0\rangle$$

$$= \sum_n \frac{1}{(n-1)!}\phi_i \left(\sum_j \phi_j a_j^\dagger\right)^{n-1}|0\rangle = \phi_i \sum_{n=1}^{\infty} \frac{1}{(n-1)!}\left(\sum_j \phi_j a_j^\dagger\right)^{n-1}|0\rangle$$

$$= \phi_i \exp\left[\sum_j \phi_j a_j^\dagger\right]|0\rangle = \phi_i|\phi\rangle \quad ①$$

$\boxed{\text{Further properties}}$

1°. $\langle\phi| a_i^\dagger = \langle\phi|\bar{\phi}_i$ 　　取①的厄米共轭即可

2°. $a_i^\dagger|\phi\rangle = \partial_{\phi_i}|\phi\rangle$

证明：$a_i^\dagger \exp\left[\sum_j \phi_j a_j^\dagger\right]|0\rangle$

$$= \frac{\partial}{\partial\phi_i}\exp\left[\sum_j \phi_j a_j^\dagger\right]|0\rangle = \partial_{\phi_i}|\phi\rangle \quad \#$$

3°. Overlap

$$\langle\theta|\phi\rangle = \langle 0|\exp\left[\sum_j \bar{\theta}_j a_j\right]|\phi\rangle = \langle 0|\exp\left[\sum_j \bar{\theta}_j \phi_j\right]|0\rangle$$

$$= \exp\left[\sum_j \bar{\theta}_j \phi_j\right] \quad ③ \; \#$$

4°. from 3°, one obtains the norm of the coherent state

$$\langle\phi|\phi\rangle = \exp\left[\sum_j \bar{\phi}_j \phi_j\right]$$

5°. The coherent states form a complete set of states in Fock space.

$$\int \prod_i \frac{d\bar{\phi}_i d\phi_i}{\pi} e^{-\sum_i \bar{\phi}_i \phi_i} |\phi\rangle\langle\phi| = 1_F \equiv \hat{B}$$

$$d\bar{\phi}_i d\phi_i = d\,\mathrm{Re}\,\phi_i \, d\,\mathrm{Im}\,\phi_i$$

证 先证明 $[\hat{B}, a_i] = [\hat{B}, a_i^+] = 0$

（因任意算符均可由 a_j, a_j^+ 组成，可判断 $\hat{B}$ 正比于常数.

$$a_j \hat{B} = \int \prod_i \frac{d\bar{\phi}_i d\phi_i}{\pi} e^{-\sum_i \bar{\phi}_i \phi_i} a_j |\phi\rangle\langle\phi|$$

$$= \int \prod_i \frac{d\bar{\phi}_i d\phi_i}{\pi} e^{-\sum_i \bar{\phi}_i \phi_i} \phi_j |\phi\rangle\langle\phi|$$

$$= -\int \prod_i \frac{d\bar{\phi}_i d\phi_i}{\pi} \frac{\partial}{\partial\bar{\phi}_j}\left[e^{-\sum_i \bar{\phi}_i \phi_i}\right] |\phi\rangle\langle\phi|$$

$$\overset{分部}{=} \int \prod_i \frac{d\bar{\phi}_i d\phi_i}{\pi} e^{-\sum_i \bar{\phi}_i \phi_i} \frac{\partial}{\partial\bar{\phi}_j}\left[|\phi\rangle\langle\phi|\right]$$

$$= \int \prod_i \frac{d\bar{\phi}_i d\phi_i}{\pi} e^{-\sum_i \bar{\phi}_i \phi_i} |\phi\rangle\langle\phi| a_j \Rightarrow [\hat{B}, a_i] = 0$$

取共轭亦可证明 $[\hat{B}, a_i^+] = 0$

这个常数可为

$$\langle 0| \int \prod_i \frac{d\phi_i\, d\bar\phi_i}{\pi} e^{-\sum_i \bar\phi_i \phi_i} \langle 0|\phi\rangle\langle\phi|0\rangle$$

用: $\int d(z,\bar z)\, e^{-\bar z \omega z} = \frac{\pi}{\omega}$

$$= \int \prod_i \frac{d\phi_i\, d\bar\phi_i}{\pi} e^{-\sum_i \bar\phi_i \phi_i} = 1$$

$\boxed{\text{Fermion case}}$

$a_i|\eta\rangle = \eta_i|\eta\rangle \Rightarrow \eta_i\eta_j = -\eta_j\eta_i$, 因 $\{a_i, a_j\} = 0$.

η_i 是 Grassman 数

1° Grassman 数的函数用泰勒函数定义

$$f(\xi_1, \cdots, \xi_k) = \sum_{n=0}^{\infty} \sum_{i_1 \cdots i_n = 1}^{k} \frac{1}{n!} \frac{\partial^n f}{\partial \xi_{i_1} \xi_{i_2} \cdots \xi_{i_n}}\Big|_{\xi=0} \xi_{i_n} \xi_{i_{n-1}} \cdots \xi_{i_1}$$

$$f(\eta) = f(0) + f'(0)\eta + \underbrace{\frac{1}{2!} f''(0)\eta^2}_{=0} + \cdots$$

$$= f(0) + f'(0)\eta$$

2°. Grassman 数的导微分

$$\partial_{\eta_i} \eta_j = \delta_{ij}$$

∂_{η_i} 与 η_j 是反对易的

$$\partial_{\eta_i} \eta_j \eta_i \overset{i \neq j}{=} -\eta_j$$

3°. Grassman 数的积分

定义 $\int d\eta_i = 0$, $\int d\eta_i \eta_i = 1$

$$\underbrace{\int d\eta f(\eta) = \int d\eta \left(f(0) + f'(0)\eta\right) = f'(0)} = \partial f(\eta)$$

$\boxed{\text{积分与微分是等价的}}$

最后规定 $\{\eta_i, a_j\} = 0$

则 fermionic coherent states 可写为

$$|\eta\rangle = \exp\left(-\sum_i \eta_i a_i^{+}\right)|0\rangle$$

证明其性质

$$a_j|\eta\rangle = a_j \exp\left(-\sum_i \eta_i a_i^{+}\right)|0\rangle$$

$$= a_j\left(1 - \sum_i \eta_i a_i^{+}\right)|0\rangle$$

$$= -a_j \sum_i \eta_i a_i^{+}|0\rangle = \sum_i \eta_i a_j a_i^{+}|0\rangle = \sum_i \eta_i \{a_j, a_i^{+}\}|0\rangle$$

$$= \eta_i|0\rangle = \eta_i\left(1 - \sum_j \eta_j a_j^{+}\right)|0\rangle \quad \times$$

$f(\eta_1, \eta_2)$

$$a_j|\eta\rangle = a_j \exp\left(-\sum_i \eta_i a_i^{+}\right)|0\rangle$$ 泰勒展开 $= f(0,0) + \left.\frac{\partial f}{\partial \eta_1}\right|_{\eta=0}\eta_1 + \left.\frac{\partial f}{\partial \eta_2}\right|_{\eta=0}\eta_2 + \left.\frac{\partial^2 f}{\partial \eta_1 \partial \eta_2}\right|_{\eta=0}\eta_2\eta_1 + \left.\frac{\partial^2 f}{\partial \eta_2 \partial \eta_1}\right|_{\eta=0}\eta_1\eta_2$

$$= a_j \exp\left(-\eta_j a_j^{+} + \sum_{i \neq j} \eta_i a_i^{+}\right)|0\rangle$$

$$= a_j\left[1 - \sum_j \eta_j a_j^{+} + \sum_{n=1} \frac{1}{n!}\frac{\partial f}{\partial \eta_{i_1} \cdots \partial \eta_{i_n}} \eta_{i_n} \cdots \eta_{i_1}\right]|0\rangle$$

$[\eta_j a_j]$ $$e^{A+B} e^{B} = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A-B,[A,B]]+\cdots}$$

$[\eta_j a_j^+, \eta_i a_i^+]$ Baker-Hausdorff 公式

$= \eta_j a_j^+ \eta_i a_i^+ - \eta_i a_i^+ \eta_j a_j^+$

$= -\eta_j \eta_i a_j^+ a_i^+ + \eta_i \eta_j a_i^+ a_j^+ = \eta_i \eta_j a_j^+ a_i^+ + \eta_i \eta_j a_i^+ a_j^+$

$= \eta_i \eta_j \{a_i^+, a_j^+\} = 0$

因此 $\boxed{e^A e^B = e^{A+B}}$

$a_j|0\rangle = a_j \exp(-\eta_j a_j^+) \exp\left(\sum_{i\neq j} \eta_i a_i^+\right)|0\rangle$

$= \eta_j \exp(-\eta_j a_j^+) \exp\left(\sum_{i\neq j} \eta_i a_i^+\right)|0\rangle$

$= \eta_j \exp\left(-\sum_i \eta_i a_i^+\right)|0\rangle = \eta_j |\eta\rangle$ 规定 $\boxed{\{\eta_i, a_i\} = 0}$

$\Uparrow$ 引进独立变量

$\boxed{\text{两个奇特之处}}$

1°. $\langle\eta| = \langle 0| \exp\left(-\sum_i a_i \bar{\eta}_i\right) = \langle 0| \exp\left(\sum_i \bar{\eta}_i a_i\right)$

2°. $\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta}\eta} = \int d\bar{\eta}\, d\eta\, (1 - \bar{\eta}\eta) = \int d\bar{\eta}\, d\eta\, \eta\bar{\eta} = 1$

不包括 π factor

几个 Gauss 积分

$\int d\bar{\eta}\, d\eta\, e^{-\bar{\eta} a \eta} = a \quad , \quad a \in \mathbb{C}$

$$\int d(\bar\phi,\phi)\, e^{-\bar\phi^T A\phi} = \det A$$ (证明：作业)

2节?

* * * *

Field integral for the quantum partition function

$$Z = \mathrm{tr}\, e^{-\beta(\hat H-\mu\hat N)} = \sum_n \langle n| e^{-\beta(\hat H-\mu\hat N)}|n\rangle$$

先构建 Z 的 path integral，再推广到关联函数

$$Z = \sum_n \langle n| e^{-\beta(\hat H-\mu\hat N)}|n\rangle$$

$$= \int d(\bar\Psi,\Psi)\, e^{-\sum_i \bar\Psi_i \Psi_i} \sum_n \langle n|\Psi\rangle\langle\Psi| e^{-\beta(\hat H-\mu\hat N)}|n\rangle$$

$$= \int d(\bar\Psi,\Psi)\, e^{-\sum_i \bar\Psi_i \Psi_i} \sum_n \langle -\Psi| e^{-\beta(\hat H-\mu\hat N)}|n\rangle\langle n|\Psi\rangle$$

$$= \int d(\bar\Psi,\Psi)\, e^{-\sum_i \bar\Psi_i \Psi_i} \langle -\Psi| e^{-\beta(\hat H-\mu\hat N)}|\Psi\rangle$$

注意 $\underset{\downarrow}{\langle n|\Psi\rangle}\langle\Psi|n\rangle = \langle -\Psi|n\rangle\langle n|\Psi\rangle$

Grassman 数

证明： $|\Psi\rangle = \exp\left(-\sum_i \Psi_i a_i^+\right)|0\rangle$

$$\langle\Psi| = \langle 0|\exp\left(\sum_i \bar\Psi_i a_i\right)$$

$$\langle\Psi|n\rangle = \langle 0|\exp\left(\sum_i \bar\Psi_i a_i\right)|n\rangle$$

$$\langle n|\Psi\rangle = \langle n|\exp\left(-\sum_i \Psi_i a_i^+\right)|0\rangle$$

$$\underbrace{\langle n|\exp\left(-\sum_i \Psi_i a_i^{+}\right)|0\rangle}_{\text{Grassman 数}}\ \underbrace{\langle 0|\exp\left(\sum_i \bar{\Psi}_i a_i\right)|n\rangle}_{\text{Grassman 数}}$$

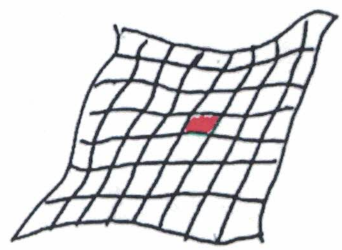
互相反对易 #

计算配分函数 (Fermion)

$$Z=\int d(\bar{\Psi},\Psi)\, e^{-\sum_i \bar{\Psi}_i \Psi_i}\langle -\Psi|e^{-\beta(\hat{H}-\mu\hat{N})}|\Psi\rangle$$

例:

$$\hat{H}(a^{+},a)=\sum_{ij} h_{ij} a_i^{+} a_j+\sum_{ijkl} V_{ijkl}\, a_i^{+} a_j^{+} a_k a_l$$



$$\delta=\frac{\beta}{N},\quad |\Psi\rangle=|\{\Psi_i\}\rangle$$

$\Psi_i(\beta)$ 是 β 的函数

$$=\int d(\bar{\Psi},\Psi)\, e^{-\sum_i \bar{\Psi}_i \Psi_i}\langle -\Psi|\boxed{\prod_{j=0}^{N} e^{-\frac{\beta}{N}(\hat{H}-\mu\hat{N})}}|\Psi\rangle$$

$$=\int d(\bar{\Psi}^0,\Psi^0)\, e^{-\sum_i \bar{\Psi}_i^0 \Psi_i^0}\langle -\Psi^0|\prod_{j=1}^{N} e^{-\frac{\beta}{N}(\hat{H}-\mu\hat{N})}|\Psi^0\rangle$$

$(-\Psi^0 \equiv \Psi^\beta$

$$=\int d(\bar{\Psi}^0,\Psi^0)\, e^{-\sum_i \bar{\Psi}_i^0 \Psi_i^0}\langle \Psi^\beta|\prod_{j=1}^{N} e^{-\frac{\beta}{N}(\hat{H}-\mu\hat{N})}|\Psi^0\rangle$$

$$=\int d(\bar{\Psi}^0,\Psi^0)\int d(\bar{\Psi},\Psi)\, e^{-\sum_i \bar{\Psi}_i^0 \Psi_i^0}\langle \Psi^\beta|e^{-\frac{\beta}{N}(\hat{H}-\mu\hat{N})}\cdots e^{-\frac{\beta}{N}(\hat{H}-\mu\hat{N})}|\Psi\rangle$$

插 N-1 个基

$$=\int d(\bar{\Psi}^0,\Psi^0)\prod_{j=1}^{N-1}\int d(\bar{\Psi}^j,\Psi^j)\, e^{-\sum_i \bar{\Psi}_i^0 \Psi_i^0}\langle \Psi^\beta|e^{-\frac{\beta}{N}(\hat{H}-\mu\hat{N})}|\Psi^{N-1}\rangle$$

$$\times\langle \Psi^{N-1}|e^{-\frac{\beta}{N}(\hat{H}-\mu\hat{N})}|\Psi^{N-2}\rangle\cdots|\Psi^1\rangle\langle \Psi^1|e^{-\frac{\beta}{N}(\hat{H}-\mu\hat{N})}|\Psi^0\rangle\, e^{-\sum_{j=1}^{N-1}\bar{\Psi}^j\Psi^j}$$

计算: $\langle \bar{\Psi}^{j} | e^{-\frac{\beta}{N}(\hat{H}-\mu\hat{N})} | \Psi^{j-1} \rangle$

$$= e^{-\frac{\beta}{N}\left[H(\bar{\Psi}^{j}, \Psi^{j-1}) - \mu N(\bar{\Psi}^{j}, \Psi^{j-1})\right]} \langle \bar{\Psi}^{j} | \Psi^{j-1} \rangle$$

$$= e^{-\frac{\beta}{N}\left[H(\bar{\Psi}^{j}, \Psi^{j-1}) - \mu N(\bar{\Psi}^{j}, \Psi^{j-1})\right]} e^{\bar{\Psi}^{j}\Psi^{j-1}}$$

$$Z = \int \prod_{j=1}^{N} d(\bar{\Psi}^{j}, \Psi^{j})\, e^{-\sum_{j=0}^{N-1} \bar{\Psi}^{j}\Psi^{j} + \sum_{j=0}^{N-1} \bar{\Psi}^{j+1}\Psi^{j}}\, e^{-\frac{\beta}{N}\sum_{j=1}^{N}\left[H(\bar{\Psi}^{j}, \Psi^{j-1}) - \mu N(\bar{\Psi}^{j}, \Psi^{j-1})\right]}$$

$$= \int \prod_{j=1}^{N} d(\bar{\Psi}^{j}, \Psi^{j})\, e^{-\delta \sum_{j=0}^{N-1}\left[\delta^{-1}(\bar{\Psi}^{j} - \bar{\Psi}^{j+1})\cdot\Psi^{j} + H(\bar{\Psi}^{j+1}, \Psi^{j}) - \mu N(\bar{\Psi}^{j+1}, \Psi^{j})\right]}$$

$$= \int_{\substack{\bar{\Psi}^{0} = -\bar{\Psi}^{N} \\ \Psi^{0} = -\Psi^{N}}} \prod_{j=1}^{N} d(\bar{\Psi}^{j}, \Psi^{j})\, e^{-\delta \sum_{j=0}^{N-1}\left[(\partial_\tau \bar{\Psi}^{j})\Psi^{j} + H(\bar{\Psi}^{j+1}, \Psi^{j}) - \mu N(\bar{\Psi}^{j+1}, \Psi^{j})\right]}$$

$$= \int_{\substack{\bar{\Psi}(0) = -\bar{\Psi}(\beta) \\ \Psi(0) = -\Psi(\beta)}} \overbrace{\boxed{\prod_{j=1}^{N} d(\bar{\Psi}^{j}, \Psi^{j})}}^{\mathcal{D}(\bar{\Psi}, \Psi)}\, e^{-\int_0^\beta d\tau\left[\overbrace{\underline{(\partial_\tau \bar{\Psi})\Psi}}^{\text{分部}} + H(\bar{\Psi}, \Psi) - \mu N(\bar{\Psi}, \Psi)\right]}$$

$$= \int_{\substack{\bar{\Psi}(0) = -\bar{\Psi}(\beta) \\ \Psi(0) = -\Psi(\beta)}} \mathcal{D}(\bar{\Psi}, \Psi)\, e^{-\underbrace{\int_0^\beta d\tau\left[\bar{\Psi}\partial_\tau\Psi + H(\bar{\Psi}, \Psi) - \mu N(\bar{\Psi}, \Psi)\right]}_{S(\bar{\Psi}, \Psi)}}$$

应用例: Plasma theory of the interacting electron gas

$$\hat{H}(a^{+}, a) = \sum_{ij} h_{ij} a_i^{+} a_j + \sum_{ijkl} V_{ijkl} a_i^{+} a_j^{+} a_k a_l$$

$$S = \int_0^\beta d\tau \left\{\sum_{ij}\left[\bar{\Psi}_i(\tau)(\partial_\tau - \mu)\delta_{ij} + h_{ij}\right]\Psi_j(\tau) + \sum_{ijkl} V_{ijkl} \bar{\Psi}_i(\tau)\bar{\Psi}_j(\tau)\Psi_k(\tau)\Psi_l(\tau)\right\}$$

作业: 利用 $\frac{x}{\sin x} = \prod_{n=1}^{\infty} \frac{1}{1 - \frac{x^2}{\pi^2 n^2}}$ 和路径积分, 计算谐振子 $\hat{H} = \hbar\omega(\hat{a}^{+}\hat{a} + \frac{1}{2})$ 的 Z

对于相互作用电子气

~~$\hat{H}(a^+,a) = \sum_{ij} h(i-j) a_i^+ a_j + \sum$~~

$$\hat{H}(a^+,a) = \sum_{\vec{k}} \varepsilon_{\vec{k}} a^+_{\vec{k}\sigma} a_{\vec{k}\sigma} + \frac{1}{2}\sum_{\substack{\vec{k}\vec{k}'\\ \sigma\sigma'}} V(\vec{q})\, a^+_{\vec{k}+\vec{q}\sigma} a^+_{\vec{k}'-\vec{q}\sigma'} a_{\vec{k}'\sigma'} a_{\vec{k}\sigma}$$

$$S = \int_0^\beta d\tau \Big[\sum_{\vec{k}\sigma} \bar{\Psi}_{\vec{k}\sigma}(\tau)(\partial_\tau - \mu + \varepsilon_{\vec{k}\sigma})\Psi_{\vec{k}\sigma}(\tau) + \frac{1}{2}\sum_{\substack{\vec{k}\vec{k}'\vec{q}\\ \sigma\sigma'}} V(\vec{q})\, \bar{\Psi}_{\vec{k}+\vec{q}\sigma}(\tau) \bar{\Psi}_{\vec{k}'-\vec{q}\sigma'}(\tau)$$

$$\times \Psi_{\vec{k}'\sigma'}(\tau) \Psi_{\vec{k}\sigma}(\tau) \Big]$$

变换到 Matsubara 表象

$$\Psi(\tau) = \frac{1}{\sqrt{\beta}} \sum_{\omega_n} \Psi_n e^{-i\omega_n \tau}, \quad \omega_n = \begin{cases} 2n\pi T \to \boxed{2n\pi/\beta} & \text{bosons} \\ (2n+1)\pi T \to \boxed{(2n+1)\pi/\beta} & \text{fermions} \end{cases} \quad n \in \mathbb{Z}$$

$$S = \int_0^\beta d\tau \Big[\sum_{\vec{k}\sigma} \sum_{nn'} \frac{1}{\beta} \bar{\Psi}_{\vec{k}\sigma n} e^{i\omega_n \tau} (\partial_\tau - \mu + \varepsilon_{\vec{k}\sigma}) \Psi_{\vec{k}\sigma n'} e^{-i\omega_{n'}\tau}$$

$$+ \frac{1}{2} \sum_{\substack{\vec{k}\vec{k}'\vec{q}\\ \sigma\sigma'}} \sum_{\substack{nn'\\ n_1 n_1'}} V(\vec{q}) \frac{1}{\beta^2} \bar{\Psi}_{\vec{k}+\vec{q}\sigma,n} e^{i\omega_n\tau} \bar{\Psi}_{\vec{k}'-\vec{q}\sigma' n'} e^{i\omega_{n'}\tau} \Psi_{\vec{k}'\sigma' n_1} e^{-i\omega_{n_1}\tau}$$

$$\times \Psi_{\vec{k}\sigma n_1'} e^{-i\omega_{n_1'}\tau} \Big]$$

$$= \int_0^\beta d\tau \Big[\sum_{\vec{k}\sigma} \sum_{nn'} \frac{1}{\beta} \bar{\Psi}_{\vec{k}\sigma n} (-i\omega_{n'} - \mu + \varepsilon_{\vec{k}\sigma}) \Psi_{\vec{k}\sigma n'} e^{i(\omega_n - \omega_{n'})\tau}$$

$$+ \frac{1}{2} \sum_{\substack{\vec{k}\vec{k}'\vec{q}\\ \sigma\sigma'}} \sum_{\substack{nn'\\ n_1 n_1'}} \frac{1}{\beta^2} V(\vec{q}) \bar{\Psi}_{\vec{k}+\vec{q}\sigma,n} \bar{\Psi}_{\vec{k}'-\vec{q}\sigma' n'} \Psi_{\vec{k}'\sigma' n_1} \Psi_{\vec{k}\sigma n_1'} e^{i(\omega_n + \omega_{n'} - \omega_{n_1} - \omega_{n_1'})\tau}$$

$$\boxed{\int_0^\beta d\tau\, e^{-i\omega_n \tau} = \beta \delta_{n,0}}$$

作业：利用路径积分计算自由电子气Pauli抗磁性

$$\hat{H}_z = -\mu_0 \vec{B}\cdot\vec{S}$$

计算自由能F，并计算零温极限下 $\chi = -\frac{\partial^2 F}{\partial B^2}\Big|_0$

$$S=\sum_{\vec{k}\alpha n}\bar{\Psi}_{\vec{k}\alpha n}(-i\omega_n-\mu+\varepsilon_{\vec{k}\alpha})\Psi_{\vec{k}\alpha n}+\frac{1}{2\beta}\sum_{\substack{\vec{k}\vec{k}'\vec{q}\\ \alpha\alpha'}}\sum_{\substack{nn'\\ n_1n_1'}}V(\vec{q})\bar{\Psi}_{\vec{k}+\vec{q}\alpha,n}\bar{\Psi}_{\vec{k}'-\vec{q}\alpha',n'}$$

$$\times\Psi_{\vec{k}'\alpha'n_1}\Psi_{\vec{k}\alpha n_1'}\delta_{n+n',n_1+n_1'}$$

引进 4-动量 $p\equiv(\vec{p},\omega_n)$

$$S=\sum_{k\alpha,n}\bar{\Psi}_{k\alpha}\left(-i\omega_n+\frac{\vec{k}^2}{2m}-\mu\right)\Psi_{k\alpha}+\frac{1}{2\beta}\sum_{\substack{kk'q\\ \alpha\alpha'}}V(\vec{q})\bar{\Psi}_{k+q\alpha}\bar{\Psi}_{k'-q\alpha'}\Psi_{k'\alpha'}\Psi_{k\alpha}$$

自由电子气的 Green 函数（动量-频率空间）

$$\langle\cdots\rangle=\frac{\int\mathcal{D}(\bar{\Psi},\Psi)\,e^{-S[\bar{\Psi},\Psi]}(\cdots)}{\int\mathcal{D}(\bar{\Psi},\Psi)\,e^{-S[\bar{\Psi},\Psi]}}$$

$$\langle\Psi_{p\alpha}\bar{\Psi}_{p\alpha}\rangle_0=\frac{\int\mathcal{D}(\bar{\Psi},\Psi)\,\Psi_{p\alpha}\bar{\Psi}_{p\alpha}\,e^{-\sum_{k\alpha'}\bar{\Psi}_{k\alpha'}(-i\omega_n+\frac{k^2}{2m}-\mu)\Psi_{k\alpha'}}}{\int\mathcal{D}(\bar{\Psi},\Psi)\,e^{-\sum_{k\alpha'}\bar{\Psi}_{k\alpha'}(-i\omega_n+\frac{k^2}{2m}-\mu)\Psi_{k\alpha'}}}$$

首先算下

$$\int\mathcal{D}(\bar{\Psi},\Psi)\,e^{-\sum_{k\alpha'}\bar{\Psi}_{k\alpha'}(-i\omega_n+\frac{k^2}{2m}-\mu_{\alpha'})\Psi_{k\alpha'}}$$

$$=\int\prod_{k\alpha'}d(\bar{\Psi}_{k\alpha'},\Psi_{k\alpha'})\,e^{-\sum_{k\alpha'}\bar{\Psi}_{k\alpha'}(-i\omega_n+\frac{k^2}{2m}-\mu_{\alpha'})\Psi_{k\alpha'}}$$

$$\left(\int d(\bar{\Psi},\Psi)\,e^{-\bar{\Psi}a\Psi}=a\right.$$

$$=\underline{M\prod_{k\alpha'}\left(-i\omega_n+\frac{k^2}{2m}-\mu_{\alpha'}\right)}$$

<u>这个可以严格计算</u>

写产生函数分法算传播子

Grassman 数

$$\int \mathcal{D}(\bar\Psi,\Psi)\, \Psi_{p\alpha} \bar\Psi_{p\alpha}\, e^{-\sum_{k\alpha'} \bar\Psi_{k\alpha'}(-i\omega_n + \frac{k^2}{2m} - \mu)\Psi_{k\alpha'} + \sum_{k\alpha'} \bar J_{k\alpha'}\Psi_{k\alpha'} + \sum_{k\alpha'} \bar\Psi_{k\alpha'} J_{k\alpha'}}$$

$\boxed{分子}$

$$= \frac{\delta^2}{\delta \bar J_{-p\alpha}\, \delta J_{-p\alpha}} \int \mathcal{D}(\bar\Psi,\Psi)\, e^{-\sum_{k\alpha'} \bar\Psi_{k\alpha'}(-i\omega_n + \frac{k^2}{2m}-\mu)\Psi_{k\alpha'} + \sum_{k\alpha'} (\bar J_{-k\alpha'}\Psi_{k\alpha'} + \bar\Psi_{k\alpha'} J_{-k\alpha'})} \Bigg|_{\substack{J=0\\ \bar J=0}}$$

$$= \frac{\delta^2}{\delta \bar J_{-p\alpha}\, \delta \bar J_{-p\alpha}} \left[\prod_{k\alpha'} \left(-i\omega_n + \frac{k^2}{2m} - \mu_{\alpha'}\right) e^{\bar J_{-k\alpha'} \frac{1}{-i\omega_n + \frac{k^2}{2m} - \mu} J_{-k\alpha'}} \right] \Bigg|_{\substack{J=0\\ \bar J=0}}$$

$$= \frac{1}{-i\omega_n + \frac{k^2}{2m} - \mu} \prod_{k\alpha'} \left(-i\omega_n + \frac{k^2}{2m} - \mu_{\alpha'}\right)$$

知道 Z，可以求出 F

$F = -\frac{1}{\beta}\ln Z$

作业：已知 玻色子 $\hat H = \sum_\xi \omega_\xi \hat a^\dagger_\xi \hat a_\xi$，2 章课 求 Green function

因此传播子

$$\langle \Psi_{p\alpha} \bar\Psi_{p\alpha} \rangle_0 = \frac{1}{-i\omega_n + \frac{k^2}{2m} - \mu_\alpha}$$

Green function 为负 $-\langle \Psi_{p\alpha} \bar\Psi_{p\alpha} \rangle$

Hubbard-Stratonovich transformation in electron gases

处理相互作用项！

Generally, $S_{int} = V_{\alpha\beta\gamma\delta}\, \bar\Psi_\alpha \Psi_\beta \bar\Psi_\gamma \Psi_\delta$

引进复合"算符" $\hat\rho_{\alpha\beta} = \bar\Psi_\alpha \Psi_\beta$

$S_{int} = V_{\alpha\beta\gamma\delta}\, \hat\rho_{\alpha\beta}\, \hat\rho_{\gamma\delta}$

令 $m \equiv \alpha\beta,\ n \equiv \gamma\delta$, $S_{int} = V_{mn}\, \hat\rho_m \hat\rho_n$ 双线性形式

$$e^{-\hat{\rho}_m V_{mn}\hat{\rho}_n} = \underbrace{\int \mathcal{D}\phi\, e^{-\frac{1}{4}\phi_m (V^{-1})_{mn}\phi_n}}_{1} \cdot e^{-\hat{\rho}_m V_{mn}\hat{\rho}_n}$$

(差归一化常数)

ϕ_m, ϕ_n 是 玻色场

$$= \int \mathcal{D}\phi\, \exp\left[-\frac{1}{4}\phi_m (V^{-1})_{mn}\phi_n - \hat{\rho}_m V_{mn}\hat{\rho}_n\right]$$ ，相同指标表求和

$\phi_m \to \phi_m + 2i V_{mp}\hat{\rho}_p$

$$= \int \mathcal{D}\phi\, \exp\left[-\frac{1}{4}(\phi_m \pm 2i V_{mp}\hat{\rho}_p)(V^{-1})_{mn}(\phi_n + 2i V_{nq}\hat{\rho}_q) - \hat{\rho}_m V_{mn}\hat{\rho}_n\right]$$

$$= \int \mathcal{D}\phi\, \exp\left[-\frac{1}{4}(\phi_m V^{-1}_{mn} \pm 2i V_{mp}\hat{\rho}_p V^{-1}_{mn})(\phi_n + 2i V_{nq}\hat{\rho}_q) - \hat{\rho}_m V_{mn}\hat{\rho}_n\right]$$

$$= \int \mathcal{D}\phi\, \exp\left[-\frac{1}{4}(\phi_m V^{-1}_{mn}\phi_n + 2i\phi_m V^{-1}_{mn} V_{nq}\hat{\rho}_q \pm 2i V_{mp}\hat{\rho}_p V^{-1}_{mn}\phi_n \mp 4 V_{mp}\hat{\rho}_p V^{-1}_{mn} V_{nq}\hat{\rho}_q) - \hat{\rho}_m V_{mn}\hat{\rho}_n\right]$$

利用 $(V^{-1})_{mn}V_{nq} = \delta_{mq}$，$(V^{-1})_{mn}V_{mp} \overset{对称}{=} \delta_{np}$

$$= \int \mathcal{D}\phi\, \exp\left[-\frac{1}{4}(\underbrace{\phi_m V^{-1}_{mn}\phi_n + 2i\phi_m\delta_{mq}\hat{\rho}_q \pm 2i\hat{\rho}_p\delta_{np}\phi_n}_{2i\phi_m\hat{\rho}_m + 2i\hat{\rho}_n\phi_n} \mp \underbrace{4\hat{\rho}_p\delta_{np}V_{nq}\hat{\rho}_q}_{\hat{\rho}_p V_{pq}\hat{\rho}_q}) - \hat{\rho}_m V_{mn}\hat{\rho}_n\right]$$

$$= \int \mathcal{D}\phi\, \exp\left[-\frac{1}{4}(\phi_m V^{-1}_{mn}\phi_n \overset{+4i\hat{\rho}_m\phi_m}{\cancel{+4i\phi_m\hat{\rho}_m}} \mp 4\hat{\rho}_p V_{pq}\hat{\rho}_q) - \hat{\rho}_m V_{mn}\hat{\rho}_n\right]$$

$$= \int \mathcal{D}\phi\, \exp\left[-\frac{1}{4}\phi_m V^{-1}_{mn}\phi_n - i\phi_m\hat{\rho}_m\right]$$ 额外用交标差项可以全部消掉

Different channels

$\bar{\Psi}_\alpha \Psi_\beta \bar{\Psi}_\gamma \Psi_\delta$

$(\alpha\beta)(\gamma\delta)$: density channel

$(\alpha\gamma)(\beta\delta)$: Cooper channel

$(\alpha\delta)(\beta\gamma)$: exchange channel

处理 plasma

$$S[\Psi,\bar{\Psi}] = \sum_p \bar{\Psi}_{p\sigma}\left(-i\omega_n + \frac{\vec{p}^2}{2m} - \mu\right)\Psi_{p\sigma} + \frac{1}{2\beta}\sum_{\substack{pp'q\\ \sigma\sigma'}} \bar{\Psi}_{p+q\sigma}\bar{\Psi}_{p'-q\sigma'} \underset{V(\vec{q})}{\overset{\frac{4\pi e^2}{|\vec{q}|^2}}{\parallel}}$$

$$\times \Psi_{p'\sigma'}\Psi_{p\sigma}$$

相互作用项：

$$S_{int} = \sum_{\substack{pp'q\\ \sigma\sigma'}} \bar{\Psi}_{p+q\sigma}\Psi_{p\sigma}\left[\frac{1}{2\beta}V(\vec{q})\right]\bar{\Psi}_{p'-q\sigma'}\Psi_{p'\sigma'}$$

$$= \sum_q\left(\sum_{p\sigma}\bar{\Psi}_{p+q\sigma}\Psi_{p\sigma}\right)\left[\frac{1}{2\beta}V(\vec{q})\right]\left[\sum_{p'\sigma'}\bar{\Psi}_{p'-q\sigma'}\Psi_{p'\sigma'}\right]$$

这里 $\rho_q = \sum_{p\sigma}\bar{\Psi}_{p+q\sigma}\Psi_{p\sigma}$，显然是密度算符的傅氏分量

$$= \sum_q \rho_q\left[\frac{1}{2\beta}V(\vec{q})\right]\rho_{-q}$$

$$1 \equiv \int \mathcal{D}\phi \exp\left[-\frac{e^2}{4}2\beta\,\phi_q V^{-1}(\vec{q})\phi_{-q}\right]$$

$$= \int \mathcal{D}\phi \exp\left[-\frac{e^2\beta}{2}\phi_q V^{-1}(\vec{q})\phi_{-q}\right]$$

$$\phi_q \to \phi_q + \frac{i}{e} V(q) \frac{\delta_q}{\beta}$$

$$S[\phi, \bar{\Psi}_\sigma, \Psi_\sigma]$$

$$= \frac{\beta}{8\pi} \sum_q \phi_q \vec{q}^2 \phi_{-q} + \sum_{pp'} \bar{\Psi}_{p\sigma} \left[\left(-i\omega_n + \frac{\vec{p}^2}{2m} - \mu \right) \delta_{pp'} + ie\phi_{p'-p} \right] \Psi_{p'\sigma}$$

两部分对易

即新的作用量

积掉费米场自由度

$$Z = \int \mathcal{D}(\bar{\Psi}, \Psi) \mathcal{D}\phi \exp\left[-\frac{\beta}{8\pi} \sum_q \phi_q \vec{q}^2 \phi_{-q} \right] \exp\left\{ -\sum_{pp'} \bar{\Psi}_{p\sigma} \left[\left(-i\omega_n + \frac{\vec{p}^2}{2m} - \mu \right) \delta_{pp'} \right.\right.$$

$$\left.\left. + ie\phi_{p'-p} \right] \Psi_{p'\sigma} \right\}$$

$$= \int \mathcal{D}\phi \, e^{-\frac{\beta}{8\pi} \sum_q \phi_q \vec{q}^2 \phi_{-q}} \det\left[-i\hat{\omega} + \frac{\hat{p}^2}{2m} - \mu + ie\hat{\phi} \right]$$

operator

(Re-exponentiate the determinant

$\ln \det \hat{A} = \mathrm{tr} \ln \hat{A}$

$\det \hat{A} = e^{\mathrm{tr} \ln \hat{A}}$

$$= \int \mathcal{D}\phi \, e^{-\frac{\beta}{8\pi} \sum_q \phi_q \vec{q}^2 \phi_{-q} + \mathrm{tr} \ln \left[-i\hat{\omega} + \frac{\hat{p}^2}{2m} - \mu + ie\hat{\phi} \right]}$$

则新的作用量

$$S[\phi] = \frac{\beta}{8\pi} \sum_q \phi_q \vec{q}^2 \phi_{-q} - \mathrm{tr} \ln \left[-i\hat{\omega} + \frac{\vec{p}^2}{2m} - \mu + ie\hat{\phi} \right]$$

以上结果是 exact 的

Stationary phase analysis (寻求 mean field)

$$\forall q = (\vec{q} \neq 0, \omega): \quad \frac{\delta S[\phi]}{\delta \phi_q} = 0 \quad \Rightarrow \text{saddle point}$$

Background potential

定义

$$G^{-1}[\phi] = i\omega - \frac{\hat{p}^2}{2m} + \mu - ie\hat{\phi}$$

$$\frac{\delta}{\delta\phi_q} \operatorname{tr} \ln [-G^{-1}(\phi)] = ? \quad \operatorname{tr}\left[-G(\phi)\frac{\delta}{\delta\phi_q} G^{-1}(\phi)\right]$$

$$\left(\partial_x \operatorname{tr}[f(\hat{A})] = \operatorname{tr}[f'(\hat{A})\, \partial_x \hat{A}]\right.$$

$$= \operatorname{tr}\left[-\hat{G}\frac{\delta}{\delta\phi_q}\hat{G}^{-1}\right] = -2\sum_{q_1 q_2} \hat{G}_{q_1 q_2}\left(\frac{\delta}{\delta\phi_q}\hat{G}^{-1}\right)_{q_2 q_1}$$

$$= -2\sum_{q_1, q_2} \hat{G}_{q_1 q_2}\left(-ie\,\delta_{q, q_1 - q_2}\right)$$

$$= \boxed{2ie\sum_{q_1 q_2} \hat{G}_{q_1 q_2}}\,\delta_{q, q_1 - q_2} = -2ie\sum_{q_1} \hat{G}_{q_1, q_1 - q}$$

Then the saddle equation becomes

$$\frac{\delta}{\delta\phi_q} = \frac{\beta}{4\pi}\vec{q}^{\,2}\phi_{-q} + 2ie\sum_{q_1} \hat{G}_{q_1, q_1 - q} = 0$$

$\phi_{q,\omega} = 0$ 是解

$$G_{q_1, q_1 - q} = \left[\frac{1}{i\hat{\omega} - \frac{\hat{p}^2}{2m} + \mu - ie\hat{\phi}}\right]_{q_1, q_1 - q}$$

$\boxed{\text{先看 } q \neq 0}$

$q \neq 0$ 时 $= \left[\frac{1}{i\hat{\omega} - \frac{\hat{p}^2}{2m} + \mu}\right]_{q_1, q_1 - q} = 0$

确实是解

$e\sum_{q_1} G_{q_1}$ ~忽略掉

$q=0$ 时，charge neutrality 要求 $\boxed{\phi_{q=0}=0}$

Fluctuation around $\phi_0=0$

对 $\phi=\phi-\phi_0$ 做展开，找出涨落项

$$\mathrm{tr}\ln\hat{G}^{-1}=\mathrm{tr}\ln\left[-i\hat{\omega}+\frac{\hat{p}^2}{2m}\mp\mu+ie\hat{\phi}\right]$$

$\boxed{\ln AB = \ln A+\ln B}$

$$=\mathrm{tr}\ln\left[\left(-i\omega+\frac{\hat{p}^2}{2m}\mp\mu\right)\left(1+\frac{ie\hat{\phi}}{-i\omega+\frac{\hat{p}^2}{2m}\mp\mu}\right)\right]$$

$$=\mathrm{tr}\left[\ln\left(i\omega-\frac{\hat{p}^2}{2m}+\mu\right)+\ln\left(1+\frac{ie\phi}{-i\omega+\frac{\hat{p}^2}{2m}\mp\mu}\right)\right]$$

$$=\mathrm{tr}\left[\ln G_0^{-1}+\ln\left(1+ie\hat{\phi}G_0\right)\right]$$

$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots$

$$=\mathrm{tr}\left[\ln G_0^{-1}+ie\hat{\phi}G_0+\frac{1}{2}e^2(\hat{G}_0\hat{\phi}\hat{G}_0\hat{\phi})\right]$$

$$=\underset{①}{\mathrm{tr}\ln G_0^{-1}}+\underset{②}{ie\,\mathrm{tr}[G_0\hat{\phi}]}+\underset{③}{\frac{e^2}{2}\mathrm{tr}(\hat{G}_0\hat{\phi}\hat{G}_0\hat{\phi})}$$

$G_0^{-1}=-i\omega+\frac{p^2}{2m}-\mu$

证明：作业

①项对应无相互作用情况

②项为0，因为是极值点展开

③项是要具体计算的

计算圈项

$$\frac{e^2}{2}\,\mathrm{tr}(\hat{G}_0\hat{\phi}\hat{G}_0\hat{\phi})$$

$$= \frac{e^2}{2}\sum_p (G_0)_{pq}\,\phi_{ql}\,(\hat{G}_0)_{lm}\,(\hat{\phi})_{mp}$$ 中间相同指标表求和

$$= \frac{e^2}{2}\sum_{pm} G_{0,p}\,\phi_{p,m}\,G_{0,m}\,\phi_{m,p}$$

$$= \frac{e^2}{2}\sum_{pm} G_{0,p}\,\phi_{p-m}\,G_{0,m}\,\phi_{m-p}$$

$$= \cancel{\frac{e^2}{2}\sum_{pm} G_{0,pm}\,\phi}$$ 令 $p-m=-q \Rightarrow m=p+q$

$$= \frac{e^2}{2}\sum_p G_{0,p}\,G_{0,p+q}\,\phi_q\,\phi_{-q} = \frac{e^2}{2}\beta\sum_q \Pi_q\,\phi_q\,\phi_{-q}$$

$$\Rightarrow\quad S_{eff} = \frac{\beta}{2}\sum_q \phi_q\left(\frac{q^2}{4\pi} - e^2\Pi_q\right)\phi_{-q}$$

2节课

Hubbard Model

$$H = \sum_{p\sigma} \epsilon_p C^+_{p\sigma} C_{p\sigma} + U \sum_i^N \hat{n}_{i\uparrow} \hat{n}_{i\downarrow}$$

$$= \sum_{p\sigma} \epsilon_p C^+_{p\sigma} C_{p\sigma} + U \sum_i^N C^+_{i\uparrow} C_{i\uparrow} C^+_{i\downarrow} C_{i\downarrow}$$

charge DOS　　　　spin DOS

$$H_U = \frac{U}{4} \sum_i (n_{i\uparrow} + n_{i\downarrow})^2 - \frac{U}{4} \sum_i (n_{i\uparrow} - n_{i\downarrow})^2$$

$$\triangleq -\frac{U}{4} \sum_i (n_{i\uparrow} - n_{i\downarrow})^2 = -\frac{U}{4} \sum_i (C^+_{i\uparrow} C_{i\uparrow} - C^+_{i\downarrow} C_{i\downarrow})^2$$

$$= -\frac{U}{4} \sum_i \Big(\sum_{\sigma\sigma'} C^+_{i\sigma} \sigma^z_{\sigma\sigma'} C_{i\sigma'} \Big)^2$$

写成路径积分，配分函数为

$$Z = \int \mathcal{D}(\bar{\Psi}, \Psi) \exp\Big\{ -\int_0^\beta d\tau \Big[\sum_p \bar{\Psi}_{p\sigma} (\partial_\tau + \xi_p) \Psi_{p\sigma} - \frac{U}{4} \sum_i (\bar{\Psi}_{i\sigma} \sigma^3_{\sigma\sigma'} \Psi_{i\sigma'})^2 \Big] \Big\}$$

相同指标表求和

HS 变换

$$\rho_i \equiv \bar{\Psi}_{i\sigma} \sigma^3_{\sigma\sigma'} \Psi_{i\sigma'}$$

$$1 = \int \mathcal{D}\phi \, e^{-\int_0^\beta d\tau \frac{1}{U} (\phi_i)^2} = \int \mathcal{D}\phi \, e^{-\frac{1}{U} \int_0^\beta \sum_i \phi_i^2 d\tau}$$

$$\Big(\phi_i \to \phi_i + 2 \cdot \frac{U}{4} \rho_i = \phi_i + \frac{U}{2} \rho_i$$

$$= \int \mathcal{D}\phi \exp\Big[-\frac{1}{U} \int_0^\beta \sum_i (\phi_i + \frac{U}{2} \rho_i)^2 d\tau$$

$$= \int \mathcal{D}\phi \exp\Big[-\frac{1}{U} \int_0^\beta \sum_i \big(\phi_i^2 + U \phi_i \rho_i + \frac{U^2}{4} \rho_i^2 \big) d\tau \Big]$$

代入到配分函数中

$$Z = \int \mathcal{D}(\bar{\Psi},\Psi)\mathcal{D}\phi \, \exp\left[-\int_0^\beta d\tau \sum_{\vec{p}} \bar{\Psi}_{\vec{p}\sigma}(\partial_\tau + \xi_{\vec{p}})\Psi_{\vec{p}\sigma}\right]$$

$$\times \exp\left\{-\int_0^\beta d\tau \left[\sum_i \left(\frac{1}{U}\phi_i^2 - \phi_i S_i + \frac{U}{4}S_i^2\right) - \frac{U}{4}\sum_i S_i^2\right]\right\}$$

$$= \int \mathcal{D}(\bar{\Psi},\Psi)\mathcal{D}\phi \, \exp\left[-\int_0^\beta d\tau \sum_{\vec{p}} \bar{\Psi}_{\vec{p}\sigma}(\partial_\tau + \xi_{\vec{p}})\Psi_{\vec{p}\sigma}\right]$$

$$\times \exp\left\{-\int_0^\beta d\tau \left[\sum_i \left(\frac{1}{U}\phi_i^2 + \phi_i S_i\right)\right]\right\}$$

$$= \int \mathcal{D}(\bar{\Psi},\Psi)\mathcal{D}\phi \, \exp\left[-\int_0^\beta d\tau \frac{1}{U}\sum_i \phi_i^2\right] \exp\left[-\int_0^\beta d\tau \sum_{\vec{p}} \bar{\Psi}_{\vec{p}\sigma}(\partial_\tau + \xi_{\vec{p}})\Psi_{\vec{p}\sigma}\right.$$

$$\left. ① \int_0^\beta d\tau \sum_i \bar{\Psi}_{i\sigma}\sigma^z_{\sigma\sigma'}\Psi_{i\sigma'}\phi_i\right] ①$$

写到动量空间

$$\Psi_{i\sigma} = \sum_{\vec{k}} e^{-i\vec{k}\cdot\vec{R}_i}\Psi_{\vec{k}\sigma}$$

因此 $\displaystyle\sum_i \sum_{\vec{k}\vec{k}'\vec{k}''} e^{-i\vec{k}\cdot\vec{R}_i}\bar{\Psi}_{\vec{k}\sigma}\sigma^3_{\sigma\sigma'} e^{i\vec{k}'\cdot\vec{R}_i}\Psi_{\vec{k}'\sigma'} e^{i\vec{k}''\cdot\vec{R}_i}\phi_{\vec{k}''}$

$$= \sum_{\vec{k}\vec{k}'\vec{k}''}\sum_i e^{i(\vec{k}'+\vec{k}''-\vec{k})\cdot\vec{R}_i}\bar{\Psi}_{\vec{k}\sigma}\sigma^3_{\sigma\sigma'}\Psi_{\vec{k}'\sigma'}\phi_{\vec{k}''}$$

$$= \sum_{\vec{k}\vec{k}'}\bar{\Psi}_{\vec{k}\sigma}\sigma^3_{\sigma\sigma'}\Psi_{\vec{k}'\sigma'}\phi_{\vec{k}-\vec{k}'}$$

因此e指数①式写为

$$\exp\left[-\int_0^\beta d\tau \sum_{\substack{\vec{p}\vec{p}'\\ \sigma\sigma'}} \bar{\Psi}_{\vec{p}\sigma}\left[(\partial_\tau + \xi_{\vec{p}})\delta_{\sigma\sigma'}\delta_{\vec{p}\vec{p}'} + \sigma^3_{\sigma\sigma'}\phi_{\vec{p}-\vec{p}'}\right]\Psi_{\vec{p}'\sigma'}\right]$$

变到频率空间

$$\Psi_{p\sigma} = \sum_n \frac{1}{\sqrt{\beta}} \Psi_{p\sigma} e^{-i\omega_n \tau}$$

$$\exp\left\{- \sum_{\substack{pp'\\ \sigma\sigma'}} \bar{\Psi}_{p\sigma} \left[(-i\omega_n + \xi_p)\delta_{\sigma\sigma'}\delta_{pp'} + \sigma^3_{\sigma\sigma'} \phi_{p-p'}\right] \Psi_{p'\sigma'}\right\}$$

令 $\phi_i = \frac{U}{2} m_i$，则

$$Z = \int \mathcal{D}(\bar{\Psi}, \Psi) \mathcal{D}m \exp\left[-\frac{U}{4}\int_0^\beta d\tau \sum_i m_i^2 \cancel{-} \sum_{\substack{pp'\\ \sigma\sigma'}} \bar{\Psi}_{p\sigma} \left[(-i\omega_n + \xi_p)\delta_{\sigma\sigma'}\delta_{pp'} + \frac{U}{2\sqrt{\beta}} \sigma^3_{\sigma\sigma'} m_{p-p'}\right. \Psi_{p'\sigma'}\right\}$$

?

对费米场的积分做掉

$$= \int \mathcal{D}m \exp\left[-\frac{U}{4}\int_0^\beta d\tau \sum_i m_i^2\right] \det\left[(-i\hat{\omega} + \hat{\xi}_p) + \frac{U}{2\sqrt{\beta}} \sigma^3 \hat{m}\right]$$

$$= \int \mathcal{D}m \exp\left[-\frac{U}{4}\int_0^\beta d\tau \sum_i m_i^2\right] \det\left[-\hat{G}_0^{-1} + \frac{U}{2\sqrt{\beta}} \sigma^3 \hat{m}\right]$$

$$= \int \mathcal{D}m \exp\left[-\frac{U}{4}\int_0^\beta d\tau \sum_i m_i^2\right] \det\left[-\hat{G}_0^{-1}\left(1 - \frac{U}{2\sqrt{\beta}} \sigma^3 \hat{m} \hat{G}_0\right)\right]$$

$$= Z_0 \int \mathcal{D}m \exp\left[-\frac{U}{4}\int_0^\beta d\tau \sum_i m_i^2\right] \det\left(1 - \frac{U}{2\sqrt{\beta}} \sigma^3 \hat{m} \hat{G}_0\right)$$

重新指数化

$$= Z_0 \int \mathcal{D}m \exp\left[-\frac{U}{4}\int_0^\beta d\tau \sum_i m_i^2 + \mathrm{tr} \ln\left(1 - \frac{U}{2\sqrt{\beta}} \sigma^3 \hat{m} \hat{G}_0\right)\right]$$

看看有无有限的 m 存在

对非线性项做展开（只要偶数项，奇数项为0）

$$\mathrm{tr} \ln\left(1 - \frac{U}{2} \sigma^3 \hat{m} \hat{G}_0\right) = \mathrm{tr}\left[-\frac{1}{2}\left(\frac{U}{2\sqrt{\beta}}\right)^2 \sigma_3 \hat{m} \hat{G}_0 \sigma_3 \hat{m} \hat{G}_0 - \frac{1}{4}\left(\frac{U}{2\sqrt{\beta}}\right)^4 \sigma_3 \hat{m} \hat{G}_0 \sigma_3 \hat{m} \hat{G}_0 \times \sigma_3 \hat{m} \hat{G}_0 \sigma_3 \hat{m} \hat{G}_0\right]$$

$$= -\mathrm{tr}\left[\frac{U^2}{8\beta}(\sigma_3 \hat{m} \hat{G}_0)^2 + \frac{U^4}{64\beta^2}(\sigma_3 \hat{m} \hat{G}_0)^4 \right]$$

$$\mathrm{tr}(\sigma_3 \hat{m} \hat{G}_0)^2$$

$$= \sum_{p\sigma} \langle p\sigma | \sigma_3 \hat{m} \hat{G}_0 \sigma_3 \hat{m} \hat{G}_0 | p\sigma \rangle$$

$$= \sum_{\substack{p\sigma \\ p'\sigma'}} \langle p\sigma | \sigma_3 \hat{m} \hat{G}_0 | p'\sigma' \rangle \langle p'\sigma' | \sigma_3 \hat{m} \hat{G}_0 | p\sigma \rangle$$

$\hat{m} | p''\sigma'' \rangle \langle p''\sigma'' |$

$$= \sum_{\substack{p\sigma \\ p'\sigma'}} \sigma^3_{\sigma\sigma'} m_{p-p'} \hat{G}_{0p'} \sigma^3_{\sigma'\sigma} \hat{m}_{p'-p} \hat{G}_{0p}$$

$\mathrm{tr}(\sigma^3 \sigma^3) = 2$

$(p - p' = -q \Rightarrow p' = p + q$

$$= \sum_{\substack{q\sigma\sigma' \\ p}} \sigma^3_{\sigma\sigma'} \sigma^3_{\sigma'\sigma} m_{-q} \hat{G}_{0p+q} \hat{m}_q \hat{G}_{0p}$$

$$= 2 \sum_q \Big(\sum_p \hat{G}_{0p+q} \hat{G}_{0p} \Big) |\hat{m}_q|^2$$

Linder hand 公式

同样地

$$\mathrm{tr}(\sigma_3 \hat{m} \hat{G}_0)^4$$

$$= \sum_{p\sigma} \langle p\sigma | \sigma_3 \hat{m} \hat{G}_0 \sigma_3 \hat{m} \hat{G}_0 \sigma_3 \hat{m} \hat{G}_0 \sigma_3 \hat{m} \hat{G}_0 | p\sigma \rangle$$

$$= \sum_{\substack{p\sigma \\ p'\sigma' \\ p''\sigma'' \\ p'''\sigma'''}} \langle p\sigma | \sigma_3 \hat{m} \hat{G}_0 | p'\sigma' \rangle \langle p'\sigma' | \sigma_3 \hat{m} \hat{G}_0 | p''\sigma'' \rangle \langle p''\sigma'' | \sigma_3 \hat{m} \hat{G}_0 | p'''\sigma''' \rangle \langle p'''\sigma''' | \sigma_3 \hat{m} \hat{G}_0 | p\sigma \rangle$$

$$= \sum_{\substack{p\sigma \\ p'\sigma' \\ p''\sigma'' \\ p'''\sigma'''}} (\sigma_3)_{\sigma\sigma'} m_{p-p'} G_{0p'} (\sigma_3)_{\sigma'\sigma''} m_{p'-p''} G_{0p''} (\sigma_3)_{\sigma''\sigma'''} m_{p''-p'''} G_{0,p'''} (\sigma_3)_{\sigma'''\sigma} m_{p'''-p} \times G_{0,p}$$

$$= \sum_{\substack{pp' \\ p''p'''}} Tr(\sigma_3^4)\, m_{p-p'} m_{p'-p''} m_{p''-p'''} m_{p'''-p}\, G_{0p'} G_{0p''} G_{0p'''} G_{0,p}$$

$$\sigma_3^4 = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]^2 = 1$$

Gradient expansion

$$= 2 \sum_{\substack{pp' \\ p''p'''}} m_{p-p'} m_{p'-p''} m_{p''-p'''} m_{p'''-p}\, G_{0p'} G_{0p''} G_{0p'''} G_{0p}$$

令 $p-p' = -q_1, \quad p'-p'' = -q_2, \quad p''-p''' = -q_3$

$\Rightarrow p' = p+q_1, \quad p'' = p+q_1+q_2, \quad p''' = p+q_1+q_2+q_3$

$$= 2 \sum_{\substack{pq_1 \\ q_2q_3}} m_{-q_1} m_{-q_2} m_{-q_3} m_{q_1+q_2+q_3}\, G_{0,p+q_1} G_{0,p+q_1+q_2} G_{0,p+q_1+q_2+q_3} G_{0p}$$

$$= 2 \sum_{\substack{q_1q_2 \\ q_3}} \left[\sum_p G_{0,p+q_1} G_{0,p+q_1+q_2} G_{0,p+q_1+q_2+q_3} G_{0,p} \right] m_{-q_1} m_{-q_2} m_{-q_3} m_{q_1+q_2+q_3}$$

最后处理下自由项

$$-\frac{U}{4}\int_0^\beta d\tau \sum_i m_i^2$$

$$m_i = \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{R}_i} m_{\vec{k}}$$

$$\sum_i m_i^2 = \sum_i \sum_{\vec{k}\vec{k}'} e^{i(\vec{k}+\vec{k}')\cdot\vec{R}_i} m_{\vec{k}} m_{\vec{k}'} = \sum_k m_{\vec{k}}(\tau)\, m_{-\vec{k}}(\tau)$$

$$= -\frac{U}{4}\int_0^\beta d\tau \sum_{\vec{k}} m_{\vec{k}}(\tau)\, m_{-\vec{k}}(\tau) = -\frac{U}{4}\sum_k m_k m_{-k}$$

Put them together

$$S = \frac{U}{4}\sum_{q}|m_q|^2 + \frac{U^2}{4\beta}\sum_{q}\Big(\sum_{p}\hat{G}_{0p+q}\,\hat{G}_{0p}\Big)|m_q|^2$$

$$+\frac{U^4}{32\beta^2}\sum_{q_1q_2q_3}\Big[\sum_{p}G_{0,p+q_1}\,G_{0,p+q_1+q_2}\,G_{0,p+q_1+q_2+q_3}\,G_{0,p}\Big]m_{-q_1}\,m_{-q_2}\,m_{-q_3}\,m_{q_1+q_2+q_3}$$

$$=\frac{1}{2}\sum_{q}\Big\{\underbrace{\frac{U}{2}\Big[1-U\Big(-\sum_{p}\frac{1}{\beta}G_{0p+q}\,G_{0,p}\Big)\Big]}_{V_2(q)}\Big\}|m_q|^2$$

$$+\frac{U^4}{32\beta^2}\sum_{q_1q_2q_3}\Big[\sum_{p}G_{0,p+q_1}\,G_{0,p+q_1+q_2}\,G_{0,p+q_1+q_2+q_3}\,G_{0,p}\Big]m_{-q_1}\,m_{-q_2}\,m_{-q_3}\,m_{q_1+q_2+q_3}$$

记 $\Pi_q = -\frac{1}{\beta}\sum_{p}G_{0p+q}\,G_{0p}$

记 $V_2(q) = \frac{U}{2}\big[1-U\Pi_q\big]$

$$V_4(q_i) = \frac{U^4}{8\beta}\sum_{p}G_{0,p+q_1}\,G_{0,p+q_1+q_2}\,G_{0,p+q_1+q_2+q_3}\,G_{0,p}$$

则

$$S[m] = \frac{1}{2}\sum_{q}V_2(q)|m_q|^2 + \frac{1}{4\beta}\sum_{q_1q_2q_3}V_4(q_1q_2q_3)\,m_{-q_1}\,m_{-q_2}\,m_{-q_3}\,m_{q_1+q_2+q_3}$$

这个结果还可进一步简化.

$$\Pi_q = -\sum_{\vec{k}}\frac{n_F(\epsilon_{\vec{k}})-n_F(\epsilon_{\vec{k}+\vec{q}})}{i\omega_n+\xi_{\vec{k}}-\xi_{\vec{k}+\vec{q}}}$$

认为 $q_1, q_2, q_3 \ll p$，则

$$V_4(q_i) = V_4(0) = \frac{U^4}{8\beta}\sum_{p}\big[G_{0,p}\big]^4$$

特例：

$\epsilon_{\vec{k}} = \frac{k^2}{2m}$，考虑下 Lindhard 公式

$$\Pi_{\vec{q}} = -\sum_{\vec{k}} \frac{n_F(\epsilon_{\vec{k}}) - n_F(\epsilon_{\vec{k}+\vec{q}})}{i\omega_n + \xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}}}$$

$$\omega \text{小} = -\sum_{\vec{k}} \frac{n_F(\epsilon_{\vec{k}}) - n_F(\epsilon_{\vec{k}+\vec{q}})}{(\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}})\left[1 + \frac{i\omega_n}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}}}\right]}$$

$$= -\sum_{\vec{k}} \frac{n_F(\epsilon_{\vec{k}}) - n_F(\epsilon_{\vec{k}+\vec{q}})}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}}} \left[1 - \frac{i\omega_n}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}}}\right]$$

$$= \underbrace{\boxed{-\sum_{\vec{k}} \frac{n_F(\epsilon_{\vec{k}}) - n_F(\epsilon_{\vec{k}+\vec{q}})}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}}}}}_{\Pi_{0,\vec{q}}} + \sum_{\vec{k}} \frac{n_F(\epsilon_{\vec{k}}) - n_F(\epsilon_{\vec{k}+\vec{q}})}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}}} \frac{i\omega_n}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}}}$$

其中 $\sum_{\vec{k}} \frac{n_F(\epsilon_{\vec{k}}) - n_F(\epsilon_{\vec{k}+\vec{q}})}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}}}$

$$\xi_{\vec{k}+\vec{q}} = \frac{(\vec{k}+\vec{q})^2}{2m} = \frac{\vec{k}^2 + 2\vec{k}\cdot\vec{q} + \vec{q}^2}{2m} = \frac{\vec{k}^2}{2m} + \frac{\vec{k}\cdot\vec{q}}{m} + \frac{\vec{q}^2}{2m}$$

$$= \sum_{\vec{k}} \frac{n_F(\epsilon_{\vec{k}}) - n_F\left(\epsilon_{\vec{k}} + \frac{\vec{k}\cdot\vec{q}}{m} + \frac{\vec{q}^2}{2m}\right)}{\xi_{\vec{k}} - \xi_{\vec{k}} - \frac{\vec{k}\cdot\vec{q}}{m} - \frac{\vec{q}^2}{2m}}$$

$$= \sum_{\vec{k}} \frac{n_F(\epsilon_{\vec{k}}) - n_F(\epsilon_{\vec{k}}) - \frac{\partial n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}}\left(\frac{\vec{k}\cdot\vec{q}}{m} + \frac{\vec{q}^2}{2m}\right) - \frac{1}{2}\frac{\partial^2 n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}^2}\left(\frac{\vec{k}\cdot\vec{q}}{m} + \frac{\vec{q}^2}{2m}\right)^2}{-\left(\frac{\vec{k}\cdot\vec{q}}{m} + \frac{\vec{q}^2}{2m}\right)}$$

$\vec{k}$ 的奇函数

$$= \sum_{\vec{k}} \left[\frac{\partial n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}} + \frac{1}{2}\frac{\partial^2 n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}^2}\left(\cancel{\frac{\vec{k}\cdot\vec{q}}{m}} + \frac{\vec{q}^2}{2m}\right) + \cdots\right]$$

$$= \sum_{\vec{k}} \left[\frac{\partial n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}} + \frac{1}{4m}\frac{\partial^2 n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}^2}\cdot \vec{q}^2 + \cdots\right]$$

$$\simeq \gamma \int_0^\infty d\epsilon_{\vec{k}} \left[\frac{\partial n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}} + \frac{1}{4m} \frac{\partial^2 n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}^2} \vec{q}^{\,2} \right]$$

$$= \gamma \left[-1 + \underbrace{\left(\frac{1}{4m} \int_0^\infty d\epsilon_{\vec{k}} \frac{\partial^2 n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}^2} \right)}_{\xi^2} \vec{q}^{\,2} \right]$$

因此 $\Pi_{0,\vec{q}} \simeq \gamma \left[1 - \xi^2 \vec{q}^{\,2} \right]$

$$\Pi_{\vec{q}} = \gamma \left[1 - \xi^2 \vec{q}^{\,2} \right] + \sum_{\vec{k}} \frac{\partial n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}} \frac{i\omega_n}{\xi_{\vec{k}} - \xi_{\vec{k}+\vec{q}}}$$

$\cancel{i\omega_n \to \omega + i0^+}$

$$= \gamma \left[1 - \xi^2 \vec{q}^{\,2} \right] + \boxed{\int \frac{d\Omega_{\vec{k}}\, k^2 dk}{(2\pi)^d} \frac{\partial n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}} \frac{i\omega_n}{-\vec{k}\cdot\vec{q}/m}}$$

$$\int \frac{d\Omega_{\vec{k}}\, k^2 dk}{(2\pi)^{d3}} = \int \frac{d\Omega_{\vec{k}}}{(2\pi)^{d3}} k\, d\tfrac{1}{2}k^2 = \frac{1}{2\pi^2} \frac{m^*}{\hbar^2} \int d \frac{\hbar^2 k^2}{2m^*}$$

γ $\sqrt{2m\epsilon_k}$

$\frac{k^2}{2m} = \epsilon_k$

$k = \sqrt{2m\epsilon_k}$

$$\int \frac{\sin\theta_{\vec{k}}\, d\theta_{\vec{k}}\, d\varphi_{\vec{k}}}{(2\pi)^3} k^2 dk \frac{\partial n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}} \frac{i\omega_n}{-kq\cos\theta_{\vec{k}}/m} \qquad \theta_{\vec{k}} \in [0, \pi)$$

$$= \frac{1}{(2\pi)^2} \int_{+1}^{+1} d\cos\theta_{\vec{k}}\, k\,dk \frac{\partial n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}} \frac{i\omega_n}{q\cos\theta_{\vec{k}}/m}$$

$$= \frac{1}{(2\pi)^2} m^* \int_0^\infty d\epsilon_{\vec{k}} \frac{\partial n_F(\epsilon_{\vec{k}})}{\partial \epsilon_{\vec{k}}} \int_1^{-1} \frac{i\omega_n}{q\cos\theta_{\vec{k}}/m} d\cos\theta_{\vec{k}}$$

$$= -\frac{1}{(2\pi)^2} m^* \int_1^{-1} \frac{i\omega_n}{q\cos\theta_{\vec{k}}/m} d\cos\theta_{\vec{k}} = -\overset{\gamma}{\frac{m^* k_F}{2\pi^2}} \frac{1}{2k_F} \int_1^{-1} \frac{i\omega_n}{q\cos\theta_{\vec{k}}/m}$$

$$= -\gamma \int_1^{-1} \frac{i\omega_n}{2q\cos\theta_F \cdot v_F + o(q^2)}\, d\cos\theta_F$$

$$= -\gamma \frac{i\omega_n}{2q\cdot v_F} \ln\left|\cos\theta_F + O(q)\right|\Big|_1^{-1}$$

$$= \gamma \frac{i\omega_n}{2q\cdot v_F} \ln\left|\frac{1+O(q)}{1-O(q)}\right|$$

$$V_4(q_1, q_2, q_3) = u$$

$$S[m] = \frac{U^2\nu}{4}\sum_q \left[r + \xi^2 q^2 + \frac{|\omega_n|}{v|q|}\right]|m_q|^2 + \frac{uT}{4}\int dx\, m^4(x)$$

$$= \frac{1}{2}\sum_q \left[r' + \xi'^2 \vec{q}^{\,2} + \frac{|\omega_n|}{\nu'|\vec{q}|}\right]|m_q|^2 + \frac{u'}{4}\int dx\, m^4(x)$$

$\boxed{\text{Stoner transition}}$

假设 m 与 q(或 x) 无关

$$\frac{S[m]}{\beta} = \frac{r'}{2}m^2 + \frac{u'}{4}m^4 \equiv S'(m)$$

$$\frac{\delta S'(m)}{\delta m} = r'm + u'm^3 = m(r' + u'm^2)$$

m 有非零值

$$r' < 0 \Rightarrow r' = U_c(1 - U_c\nu)/2 = 0 \Rightarrow U_c\nu = 1$$

Stoner transition point

Tunneling Problem and Josephson action

所有有奇异沉积的物理均是由于低能有效 action 有反常项

库仑阻塞.

态密度：$\rho(\epsilon) = tr[\delta(\epsilon - \hat{H})] = -\frac{1}{\pi} I_m\, tr[\hat{G}(\epsilon + io^+)]$

$$= -\frac{1}{\pi} I_m\, tr[\hat{G}_n]\Big|_{i\omega_n \to \epsilon + io^+}$$

$$\hat{G}(z) = \frac{1}{z - \hat{H}}$$

Free case

Tunneling DOS 根据 Green function 的这个定义计算

$$\nu(\epsilon) = -\frac{1}{\pi} I_m\, tr(\hat{G}_n)\Big|_{i\omega_n \to \epsilon + io^+}$$

interacting case

Green function

$$G_{\alpha\beta}(\tau) = \frac{1}{Z_0} \int D(\psi, \bar{\psi})\, e^{-S[\bar{\psi}, \psi]}\, \bar{\psi}_\beta(\tau)\, \psi_\alpha(0)$$

<u>有效 Action</u>

哈密顿量

$$\hat{H} = H_0 + E_c(\hat{N} - N_0)^2$$

$$= \sum_\alpha \epsilon_\alpha a_\alpha^+ a_\alpha + E_c\left(\sum_\alpha a_\alpha^+ a_\alpha - N_0\right)^2$$

Frequency summation:

费米子：（玻色子作为作业）

$$G^{(0)}(\vec{p}, ip_n) = \frac{1}{i\omega_n - \xi_p}$$

对频率求和：

$$\frac{1}{\beta}\sum_n G^{(0)}(\vec{p}, ip_n) = n_F(\xi_p)$$

一般的 $\frac{1}{\beta}\sum_n f(ip_n) = ?$

我们计算下复变函数积分

$$\oint \frac{dz}{2\pi i} f(z)\, n_F(z) \quad , \quad n_F(z) = \frac{1}{e^{\beta z}+1}$$

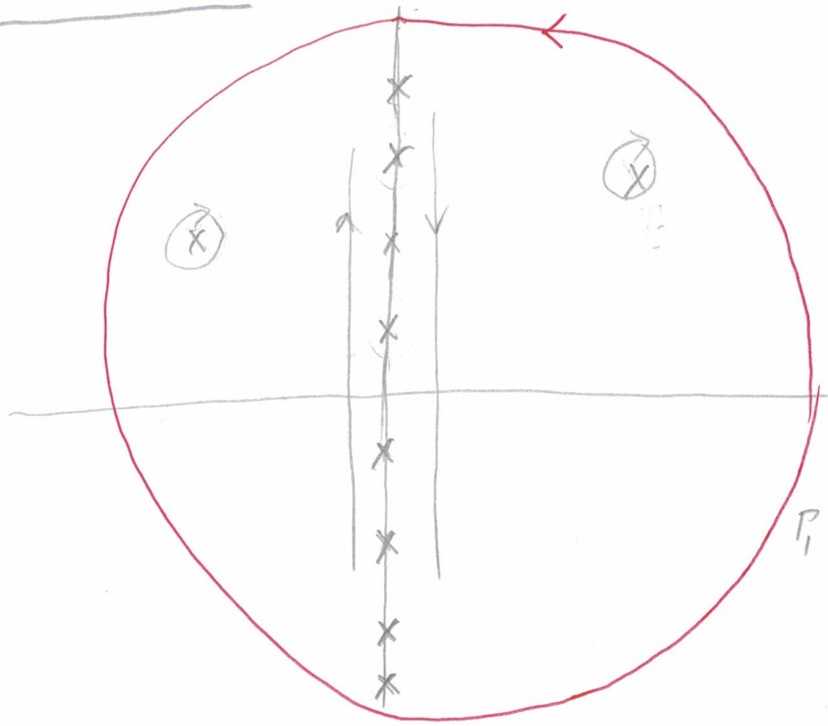
$\Longrightarrow$ $\boxed{\text{利用留数定理}}$

$f(z)$ 有奇点 z_i，有可能多个

$n_F(z)$ 奇点 $ip_n = i\,2\pi(n+1)/\beta$

$\rightarrow$ 验证：$\frac{1}{e^{ip_n\beta}+1} = \frac{1}{e^{i\,2\pi(n+1)}+1}$ $\longrightarrow \cos(2\pi(n+1)) = -1$

奇点分布绘出:



选择积分路径为无穷远的大圆，与包围各个留数的小圆

$$\oint_{\Gamma_1} + \oint_{\Gamma_S} = 0 \longrightarrow \oint_{\Gamma_S} = 0$$

$$\oint_{-\Gamma_S} = 0 \Longrightarrow \frac{1}{\beta}\sum_n f(ip_n) + \sum_i \mathrm{Res}\left(f(z)\, n_F(z)\right)\Big|_{z_i} = 0$$

$$\Longrightarrow -\frac{1}{\beta}\sum_n f(ip_n) = -\sum_i \mathrm{Res}\left(f(z)\, n_F(z)\right)\Big|_{z_i}$$

例: $\mathcal{G}^{(0)}(ip_n) = \dfrac{1}{ip_n - \xi_p}$

$$f(z) = \frac{1}{z - \xi_p}$$

$$\Longrightarrow \frac{1}{\beta}\sum_n \mathcal{G}^{(0)}(ip_n) = n_F(\xi_p)$$

$$\frac{1}{\beta}\sum_n G^{(0)}(ip_n,\vec{p})\,G^{(0)}(ip_n+i\omega_m,\vec{k})$$

$$=\frac{n_F(\xi_{\vec p})-n_F(\xi_{\vec k})}{i\omega_m+\xi_{\vec p}-\xi_{\vec k}}$$

$$f(z)=\frac{1}{z-\xi_{\vec p}}\;\frac{1}{z+i\omega_m-\xi_{\vec k}}$$

奇点 $\xi_{\vec p}$, $\xi_{\vec k}-i\omega_m$

$$\frac{1}{\beta}\sum_n G^{(0)}(ip_n,\vec{p})\,G^{(0)}(ip_n+i\omega_m,\vec{k})$$

$$=\sum_i \mathrm{Res}\left(f(z)\,n_F(z)\right)\Big|_{z=z_i}$$

$$\frac{1}{e^{\beta(\xi_{\vec k}-i\omega_m)}+1}=\frac{1}{e^{\beta\xi_{\vec k}}+1}$$

$$=n_F(\xi_{\vec p})\frac{1}{\xi_{\vec p}+i\omega_m-\xi_{\vec k}}+n_F(\xi_{\vec k}-\underbrace{i\omega_m}_{\text{玻色形}})\frac{1}{\xi_{\vec k}-i\omega_m-\xi_{\vec p}}$$

$$=\frac{n_F(\xi_{\vec p})-n_F(\xi_{\vec k})}{i\omega_m+\xi_{\vec p}-\xi_{\vec k}}$$ → 用于 plasmon 问题

作业：已知 $D^{(0)}(i\omega_m,\vec{q})=\dfrac{2\omega_{\vec q}}{\omega_m^2+\omega_{\vec q}^2}$, ω_m 玻色形

求：$-\frac{1}{\beta}\sum_m D^{(0)}(i\omega_m,\vec{q})\,G^{(0)}(ip_n+i\omega_m,\vec{p})=?\left(\dfrac{N_{\vec q}+n_F(\xi_{\vec p}\ldots}{ip_n-\xi_{\vec p}+\omega\ldots}\right.$

整理一下:

$$S_{int}[\phi] = \sum_q \phi_q \frac{2\beta}{V(\vec{q})} \phi_{-q} - \mathrm{tr}\ln\left(-i\hat{p} + \frac{\vec{p}^2}{2m} - \mu - 2i\hat{\phi}\right)$$

$$\boxed{\phi \to -\frac{e\phi}{2}}$$

$$\to \sum_q \phi_q \frac{e^2\beta}{2V(\vec{q})} \phi_{-q} - \mathrm{tr}\ln\left(-i\hat{p} + \frac{\vec{p}^2}{2m} - \mu + ie\hat{\phi}\right)$$

$$\boxed{V(\vec{q}) = \frac{4\pi e^2}{|\vec{q}|^2}} \to \sum_q \phi_q \frac{\beta|\vec{q}|^2}{8\pi} \phi_{-q}$$

$$S_{int} = \sum_{q} \left(\frac{e^2 \beta}{2V(\vec{q})} \phi_q \phi_{-q} + ie\, n_q \phi_{-q} - ie\, \rho_q \phi_{-q} \right)$$

$$S_{eff} \to \sum_{q} \phi_q \frac{e^2 \beta}{2V(\vec{q})} \phi_{-q} + ie\, n_q \phi_{-q} - \mathrm{tr} \ln \left(-i\hat{p} + \frac{\vec{p}^{\,2}}{2m} - \mu + ie \ldots \right.$$

$\Longrightarrow$ saddle point equation

$$\frac{\delta}{\delta \phi_q} = \frac{\beta}{4\pi} \vec{q}^{\,2} \phi_{-q} + ie\, n_{-q} + 2ie \sum_{q_1} \hat{G}_{q_1, q_1 - q} = 0$$

当 $\vec{q} \to 0$ 及 $q_n \to 0$ 时

$$\boxed{-2 \sum_{q_1} \hat{G}_{q_1, q_1 - q} + n_{-q} = 0} \to \boxed{\phi_{q \to 0} = 0}$$

charge neutrality.

Inclusion of background positive charge

$$\hat{H}_{int} = \int d\vec{r}\, d\vec{r}'\, V(\vec{r}-\vec{r}')\, \underbrace{\hat{\Psi}^+(\vec{r})\, \underbrace{\hat{\Psi}^+(\vec{r}')\, \hat{\Psi}(\vec{r}')}_{\rho(\vec{r}')}\, \hat{\Psi}(\vec{r})}_{\rho(\vec{r})}$$

$$\to \int d\vec{r}\, d\vec{r}'\, V(\vec{r}-\vec{r}')\, \rho(\vec{r})\, \rho(\vec{r}')$$

凝胶模型，均匀分布的正电荷

$$\to \int d\vec{r}\, d\vec{r}'\, V(\vec{r}-\vec{r}')\, \big(\rho(\vec{r}) - n_0\big)\big(\rho(\vec{r}') - n_0\big)$$

$\boxed{\frac{1}{V}\int d\vec{r}\, \rho(\vec{r}) = n_0}$ $\Longrightarrow$ 运用这一个限制.

$n_0(\vec{r})$ 不依赖于 $\vec{r}$ $\to$ $\boxed{n_{\vec{k}},\ \vec{k}=0}$

$$\to \sum_{\vec{q}} V(\vec{q}) \Big(\sum_{\vec{p}} \hat{\Psi}^+_{\vec{p}+\vec{q}}\, \hat{\Psi}_{\vec{p}} - n_{\vec{q}}\Big)\Big(\sum_{\vec{p}'} \hat{\Psi}^+_{\vec{p}'-\vec{q}}\, \hat{\Psi}_{\vec{p}'} - n_{-\vec{q}}\Big)$$

$$\to \sum_{\vec{q}}{}' V(\vec{q})\, \big(\rho_{\vec{q}} - n_{\vec{q}}\big)\big(\rho_{-\vec{q}} - n_{-\vec{q}}\big)$$

$$\frac{1}{V}\int d\vec{r} \sum_{\vec{k}} \rho_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}} = \sum_{\vec{q}} n_{\vec{q}}\, e^{i\vec{q}\cdot\vec{r}}$$

$$= \frac{1}{V}\sum_{\vec{k}} \rho_{\vec{k}} \underbrace{\int d\vec{r}\, e^{i\vec{k}\cdot\vec{r}}}_{(2\pi)^3\delta(\vec{k})} = \boxed{\rho_{\vec{k}=0} = n_0}$$

$$n_0 = \sum_{\vec{q}} n_{\vec{q}}\, e^{i\vec{q}\cdot\vec{r}}$$

$$n_{\vec{q}} = \frac{1}{V}\int d\vec{r}\, n_0\, e^{-i\vec{q}\cdot\vec{r}}$$

$$= \frac{1}{V} n_0 (2\pi)^3 \delta(\vec{q})$$

$$= n_0 \delta_{\vec{q}}$$

如何在路径积分中恢复 charge neutrality?

$$S_{int} = \sum_q \left(\sum_{p\sigma} \bar{\psi}_{p+q\sigma} \psi_{p\sigma} - n_q \right) \left[\frac{1}{2\beta} V(\vec{q}) \right] \left(\sum_{p'\sigma'} \bar{\psi}_{p'-q\sigma'} \psi_{p'\sigma'} - n_{-q} \right)$$

$\boxed{\phi_q \to \phi_q - n_q}$ 是否能够解决这一问题

$$\text{"}1\text{"} = \int \mathcal{D}\phi \exp\left[-\sum_q \frac{2\beta}{V(\vec{q})} \phi_q \phi_{-q} \right]$$

$$\Longrightarrow Z_{int} = \int \mathcal{D}\phi \exp\left[-\sum_q \frac{2\beta}{V(\vec{q})} \phi_q \phi_{-q} + \sum_q (\rho_q - n_q) \frac{V(\vec{q})}{2\beta} (\rho_{-q} - n_{-q}) \right]$$

$$\left(\phi_q \to \phi_q + \frac{i}{2\beta} V(\vec{q}) (\rho_q - n_q) \right.$$

$$= \int \mathcal{D}\phi \exp\left\{ -\sum_q \frac{2\beta}{V(\vec{q})} \left[\phi_q + \frac{i}{2\beta} V(\vec{q}) (\rho_q - n_q) \right] \left[\phi_{-q} + \frac{i}{2\beta} V(\vec{q}) (\rho_{-q} - n_{-q}) \right] \right\}$$

$$- \sum_q (\rho_q - n_q) \frac{V(\vec{q})}{2\beta} (\rho_{-q} - n_{-q}) \Big]$$

$$= \int \mathcal{D}\phi \exp\left\{ -\sum_q \left(\frac{2\beta}{V(\vec{q})} \phi_q \phi_{-q} + 2i (\rho_q - n_q) \phi_{-q} \right) \right\}$$

$$\phi \to -\frac{e\phi}{2}$$

(15分)

写出三维自由费米子在Matsubara表象下的Green函数 $G^{(0)}(ip_n, \vec{p})$，其中$\vec{p}$为费米子动量，其能谱为$\xi_{\vec{p}}$，ip_n为Matsubara频率。并利用围道积分计算对Matsubara ~~表象~~ 频率求和：

由：

$\xi_{\vec{p}}$

$\xi_{\vec{p}}$

$$\frac{1}{\beta}\sum_n G^{(0)}(ip_n, \vec{p})\, G^{(0)}(ip_n + i\omega_m, \vec{k})$$

已知玻色子Green函数 $D^{(0)}(i\omega_m, \vec{q}) = \frac{2\omega_{\vec{q}}}{\omega_m^2 + \omega_{\vec{q}}^2}$，

ω_m为玻色子Matsubara ~~表象~~ 频率，写出其形式。

~~利用 写出三维费米子在Matsubara表象下的Green函数~~

$G^{(0)}$ ~~$(ip_n, \vec{p})$，其中 $\vec{p}$ 为~~

费米子Green函数 $G^{(0)}(ip_n, \vec{p}) = \frac{1}{ip_n - \xi_{\vec{p}}}$，写出费米子Matsubara频率 ip_n。利用围道积分计算对Matsubara频率求和：

$$-\frac{1}{\beta}\sum_m D^{(0)}(i\omega_m, \vec{q})\, G^{(0)}(ip_n + i\omega_m, \vec{p})$$