零温与有限温格林函数的关系

△ 零温：$G^{R/A} = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{A(\omega')}{\omega - \omega' \pm i0_+}$

有限温：$G(\omega_n) = \int_{-\infty}^{+\infty} \frac{d\omega'}{2\pi} \frac{A(\omega')}{i\omega_n - \omega'}$

$i\omega_n \longrightarrow \begin{cases} \omega + i0_+ & (G^R) \\ \omega - i0_+ & (G^A) \end{cases}$ 解析延拓

费曼图，费曼规则，微扰论

例：电声相互作用

$$\hat{H}_I = g \int \underbrace{\overset{\text{电子}}{\psi^+_\alpha(\vec{r},\tau)\, \psi_\alpha(\vec{r},\tau)}}_{\text{电子密度算符}} \overset{\text{声子场算符}}{\phi(\vec{r},\tau)}\, d\vec{r}$$

声子场算符（忽略了极化）

$$\phi(\vec{r}) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \sqrt{\frac{\omega(\vec{k})}{2}} \left(\hat{b}_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r}} + \hat{b}^+_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{r}} \right)$$

变到相互作用表象

$$\phi(\vec{r},\tau) = \frac{1}{\sqrt{V}} \sum_{\vec{k}} \sqrt{\frac{\omega(\vec{k})}{2}} \left[\hat{b}_{\vec{k}}\, e^{i\vec{k}\cdot\vec{r} - \tau\omega(\vec{k})} + \hat{b}^+_{\vec{k}}\, e^{-i\vec{k}\cdot\vec{r} + \tau\omega(\vec{k})} \right]$$

目的，电子格林函数

$$G(\tau_1 > \tau_2) = -\frac{1}{\Xi} \mathrm{tr} \Big(e^{-\beta(\overset{\hat{H} = \hat{H}_0 + \hat{H}_{int}}{\hat{H}} - \mu\hat{N})} \underbrace{\hat{\psi}(\vec{r}_1,\tau_1)}_{\Rightarrow e^{\tau_1(\hat{H}-\mu\hat{N})} \hat{\psi}(\vec{r}_1) e^{-\tau_1(\hat{H}-\mu\hat{N})}} \hat{\psi}^+(\vec{r}_1,\tau_1) \Big)$$

海森堡表象

复习 S矩阵 $S(\tau)$

$$e^{-\tau(\hat{H}-\mu\hat{N})} = e^{-\tau(\hat{H}_0 - \mu\hat{N})} S(\tau)$$

$$= S^{-1}(\tau)\, e^{-\tau(\hat{H}_0 - \mu\hat{N})}$$

$$\frac{\partial S(\tau)}{\partial \tau} = -\hat{H}_{int} S(\tau) \longrightarrow S(\tau) = T_\tau\, e^{-\int_0^\tau \hat{H}_{int}(\tau')d\tau'}$$

$$G(\tau_1>\tau_2) = -\frac{1}{\Xi}\Big(e^{-\beta(\hat{H}_0-\mu\hat{N})}S(\beta)S^{-1}(\tau_1)\underbrace{e^{\tau_1(\hat{H}_0-\mu\hat{N})}\hat{\psi}(\vec{r}_1)e^{-\tau_1(\hat{H}_0-\mu\hat{N})}}S(\tau_1)$$

$$S^{-1}(\tau_2)\underbrace{e^{\tau_2(\hat{H}_0-\mu\hat{N})}\hat{\psi}^+(\vec{r}_2)e^{-\tau_2(\hat{H}_0-\mu\hat{N})}}S(\tau_2)\Big)$$

$$= -\frac{1}{\Xi}tr\left\{e^{-\beta(\hat{H}_0-\mu\hat{N})}S(\beta)S^{-1}(\tau_1)\hat{\psi}_0(\vec{r}_1,\tau_1)S(\tau_1)S^{-1}(\tau_2)\hat{\psi}_0^+(\vec{r}_2,\tau_2)S(\tau_2)\right\}$$

$$S(\tau_1\,\tau_2) = T_\tau e^{-\int_{\tau_1}^{\tau_2}\hat{H}_{int}(\tau')d\tau'} = S(\tau_1)S^{-1}(\tau_2)$$

$$= -\frac{1}{\Xi}tr\left\{e^{-\beta(\hat{H}_0-\mu\hat{N})}S(\beta,\tau_1)\hat{\psi}_0(\vec{r}_1,\tau_1)S(\tau_1,\tau_2)\hat{\psi}_0^+(\vec{r}_2,\tau_2)S(\tau_2,0)\right\}$$

$$\triangle = -\frac{1}{\Xi}tr\left\{e^{-\beta(\hat{H}_0-\mu\hat{N})}T_\tau(\hat{\psi}_0(\vec{r}_1,\tau_1)\hat{\psi}_0^+(\vec{r}_2,\tau)\underbrace{S(\beta,0)}_{\text{包含了相互作用}})\right\}$$

$$\Xi = tr(e^{-\beta(\hat{H}-\mu\hat{N})})$$

$$= tr\left(e^{-\beta(\hat{H}_0-\mu\hat{N})}S(\beta)\right)$$

$$\langle\langle\cdots\rangle\rangle = \frac{tr(\cdots e^{-\beta(\hat{H}_0-\mu\hat{N})})}{tr(e^{-\beta(\hat{H}_0-\mu\hat{N})})}$$

$$\boxed{G(\vec{r}_1,\tau_1,\vec{r}_2,\tau_2) = \frac{-\langle\langle T_\tau\hat{\psi}_0(\vec{r}_1,\tau_1)\hat{\psi}_0^+(\vec{r}_2,\tau_2)S(\beta)\rangle\rangle_0}{\langle\langle S(\beta)\rangle\rangle_0}}$$ 起点

以电声相互作用为例作微扰展开

$$G(\vec{p},\tau) = -\frac{\langle\langle T_\tau S(\beta)C_{\vec{p}}(\tau)C_{\vec{p}}^+(0)\rangle\rangle_0}{\underbrace{\langle\langle S(\beta)\rangle\rangle_0}}$$ → 贡献"泡泡图"，计算时当作 1

相互作用写到动量表象下

$$\hat{H}_{int} = \sum_{\vec{q}}M_{\vec{q}}(\hat{b}_{\vec{q}}+\hat{b}_{-\vec{q}}^+)\sum_{\vec{k}}\hat{C}_{\vec{k}+\vec{q}}^+\hat{C}_{\vec{k}}$$

$$S(\beta) = e^{-\int_0^\beta d\tau' H_{int}(\tau')} \approx 1 - \int_0^\beta d\tau' H_{int}(\tau') + \frac{1}{2}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \hat{H}_{int}(\tau_1)\hat{H}_{int}(\tau_2)$$

一阶项没有贡献：$\sim \langle\langle T_\tau (b+b^+) c c^+ \rangle\rangle \Rightarrow \langle\langle b \rangle\rangle = 0$

即 $G^{(1)} = 0$

$$G^{(2)}(\vec{p},\tau) = \frac{1}{2}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \langle\langle T_\tau \hat{C}_{\vec{p}}(\tau) \hat{H}_{int}(\tau_1) \hat{H}_{int}(\tau_2) \hat{C}^+_{\vec{p}}(0) \rangle\rangle$$

$$= \frac{1}{2}\int_0^\beta d\tau_1 \int_0^\beta d\tau_2 \langle\langle T_\tau \hat{C}_{\vec{p}}(\tau) \sum_{\vec{k},\vec{q}} M_{\vec{q}} (\hat{b}_{\vec{q}}(\tau_1) + \hat{b}^+_{-\vec{q}}(\tau_1)) \hat{C}^+_{\vec{k}+\vec{q}}(\tau_1) \hat{C}_{\vec{k}}(\tau_1)$$

$$\times \sum_{\vec{k}',\vec{q}'} M_{\vec{q}'} (\hat{b}_{\vec{q}'}(\tau_2) + \hat{b}^+_{-\vec{q}'}(\tau_2)) \hat{C}^+_{\vec{k}'+\vec{q}'}(\tau_2) \hat{C}_{\vec{k}'}(\tau_2) \hat{C}^+_{\vec{p}}(0) \rangle\rangle$$

$$= \frac{1}{2}\sum_{k,q}\sum_{k',q'} \int_0^\beta d\tau_1 \int_0^\beta d\tau_2 M_q M_{q'} \langle\langle T_\tau \hat{C}_{\vec{p}}(\tau) \hat{C}^+_{\vec{k}+\vec{q}}(\tau_1) \hat{C}_{\vec{k}}(\tau_1) \hat{C}^+_{\vec{k}'+\vec{q}'}(\tau_2) \hat{C}_{\vec{k}'}(\tau_2) \hat{C}^+_{\vec{p}}(0) \rangle\rangle$$

$$\times \langle\langle (\hat{b}_{\vec{q}}(\tau_1) + \hat{b}^+_{-\vec{q}}(\tau_1))(\hat{b}_{\vec{q}'}(\tau_2) + \hat{b}^+_{-\vec{q}'}(\tau_2)) \rangle\rangle_0$$

Wick 定理

$$\langle\langle AB^+CD^+ \rangle\rangle = \langle\langle AB^+ \rangle\rangle \langle\langle CD^+ \rangle\rangle + \langle\langle AD^+ \rangle\rangle \langle\langle B^+C \rangle\rangle$$

$$\langle\langle T_\tau \hat{C}_{\vec{p}}(\tau) \hat{C}^+_{\vec{k}+\vec{q}}(\tau_1) \hat{C}_{\vec{k}}(\tau_1) \hat{C}^+_{\vec{k}'+\vec{q}'}(\tau_2) \hat{C}_{\vec{k}'}(\tau_2) \hat{C}^+_{\vec{p}}(0) \rangle\rangle_0$$

$$= \langle\langle T_\tau \hat{C}_{\vec{p}}(\tau) \hat{C}^+_{\vec{k}+\vec{q}}(\tau_1) \rangle\rangle_0 \langle\langle T_\tau \hat{C}_{\vec{k}}(\tau_1) \hat{C}^+_{\vec{k}'+\vec{q}'}(\tau_2) \rangle\rangle_0 \langle\langle T_\tau \hat{C}_{\vec{k}'}(\tau_2) \hat{C}^+_{\vec{p}}(0) \rangle\rangle_0$$

$$+ \langle\langle T_\tau \hat{C}_{\vec{p}}(\tau) \hat{C}^+_{\vec{k}+\vec{q}}(\tau_1) \rangle\rangle_0 \langle\langle T_\tau \hat{C}_{\vec{k}}(\tau_1) \hat{C}^+_{\vec{p}}(0) \rangle\rangle_0 \langle\langle T_\tau \hat{C}^+_{\vec{k}'+\vec{q}'}(\tau_2) \hat{C}_{\vec{k}'}(\tau_2) \rangle\rangle_0$$

$$+ \langle\langle T_\tau \hat{C}_{\vec{p}}(\tau) \hat{C}^+_{\vec{p}}(0) \rangle\rangle_0 \langle\langle T_\tau \hat{C}^+_{\vec{k}+\vec{q}}(\tau_1) \hat{C}_{\vec{k}'}(\tau_2) \rangle\rangle_0 \langle\langle T_\tau \hat{C}_{\vec{k}}(\tau_1) \hat{C}^+_{\vec{k}'+\vec{q}'}(\tau_2) \rangle\rangle_0$$

$$+ \langle\langle T_\tau \hat{C}_{\vec{p}}(\tau) \hat{C}^+_{\vec{k}'+\vec{q}'}(\tau_2) \rangle\rangle_0 \langle\langle T_\tau \hat{C}^+_{\vec{k}+\vec{q}}(\tau_1) \hat{C}_{\vec{k}}(\tau_1) \rangle\rangle_0 \langle\langle T_\tau \hat{C}_{\vec{k}'}(\tau_2) \hat{C}^+_{\vec{p}}(0) \rangle\rangle_0$$

$$+ \langle\langle T_\tau \hat{C}_{\vec{p}}(\tau) \hat{C}^+_{\vec{k}'+\vec{q}'}(\tau_2) \rangle\rangle_0 \langle\langle T_\tau C^+_{\vec{k}+\vec{q}}(\tau_1) \hat{C}_{\vec{k}'}(\tau_2) \rangle\rangle_0 \langle\langle T_\tau \hat{C}^+_{\vec{p}}(0) \hat{C}_{\vec{k}}(\tau_1) \rangle\rangle_0$$

$$+ \langle\langle T_\tau \hat{C}_{\vec{p}}(\tau) \hat{C}^+_{\vec{p}}(0) \rangle\rangle_0 \langle\langle T_\tau \hat{C}^+_{\vec{k}+\vec{q}}(\tau_1) \hat{C}_{\vec{k}}(\tau_1) \rangle\rangle_0 \langle\langle T_\tau \hat{C}^+_{\vec{k}'+\vec{q}'}(\tau_2) \hat{C}_{\vec{k}'}(\tau_2) \rangle\rangle_0$$

$$= -G^0_{\vec p}(\tau,\tau_1)\,G^0_{\vec k}(\tau_1,\tau_2)\,G^0_{\vec k'}(\tau_2,0)\,\delta_{\vec p,\vec k+\vec q}\,\delta_{\vec k,\vec k'+\vec q'}\,\delta_{\vec k'\vec p} \quad ①$$

$$+\,G^0_{\vec p}(\tau,\tau_1)\,G^0_{\vec k}(\tau_1,0)\,G^0_{\vec k'}(\tau_2,\tau_2)\,\delta_{\vec p,\vec k+\vec q}\,\delta_{\vec k\vec p}\,\delta_{\vec k'+\vec q'\vec k'} \quad ②$$

$$+\,G^0_{\vec p}(\tau,0)\,G^0_{\vec k'}(\tau_2,\tau_1)\,G^0_{\vec k}(\tau_1,\tau_2)\,\delta_{\vec k',\vec k+\vec q}\,\delta_{\vec k,\vec k'+\vec q'} \quad ③$$

$$+\,G^0_{\vec p}(\tau,\tau_2)\,G^0_{\vec k}(\tau_1,\tau_1)\,G^0_{\vec k'}(\tau_2,0)\,\delta_{\vec p,\vec k'+\vec q'}\,\delta_{\vec k,\vec k+\vec q}\,\delta_{\vec k',\vec p} \quad ④$$

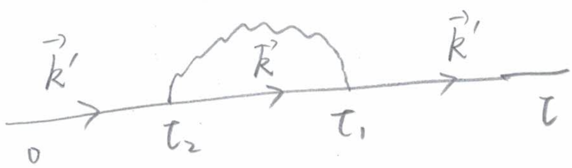
$$-\,G^0_{\vec p}(\tau,\tau_2)\,G^0_{\vec k'}(\tau_2,\tau_1)\,G^0_{\vec k}(\tau_1,0)\,\delta_{\vec p,\vec k'+\vec q'}\,\delta_{\vec k',\vec k+\vec q}\,\delta_{\vec k,\vec p} \quad ⑤$$

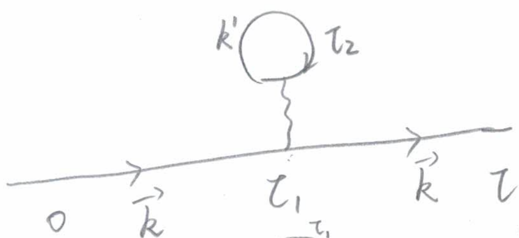
$$-\,G^0_{p}(\tau,0)\,G^0_{\vec k}(\tau_1,\tau_1)\,G^0_{\vec k'}(\tau_2,\tau_2)\,\delta_{\vec k,\vec k+\vec q}\,\delta_{\vec k',\vec k'+\vec q'} \quad ⑥$$

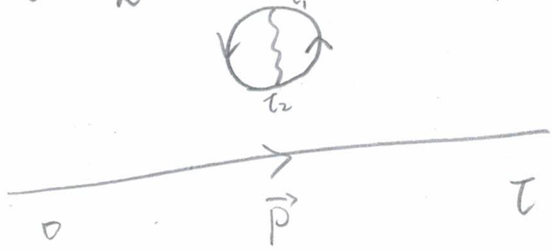
把 $\langle\langle(\hat b_{\vec q}(\tau_1)+\hat b^+_{-\vec q}(\tau_1))(\hat b_{\vec q'}(\tau_2)+\hat b^+_{\vec q'}(\tau_2))\rangle\rangle_0$ 记为 $V(\tau_1,\tau_2)$

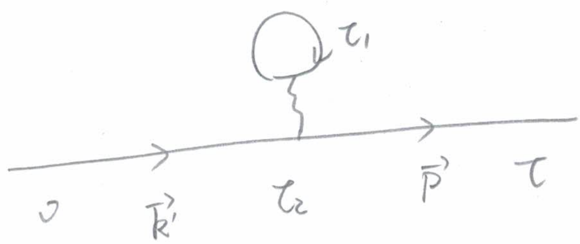
则 $\langle\langle T_\tau\,\hat C_{\vec p}(\tau)\,\hat C_{\vec k+\vec q}(\tau_1)\,\hat C_{\vec k}(\tau_1)\,\hat C^+_{\vec k+\vec q'}(\tau_2)\,\hat C_{\vec k}(\tau_2)\,\hat C^+_{\vec p}(0)\rangle\rangle_0$ $V(\tau_1,\tau_2)$ 的 6 个费曼图

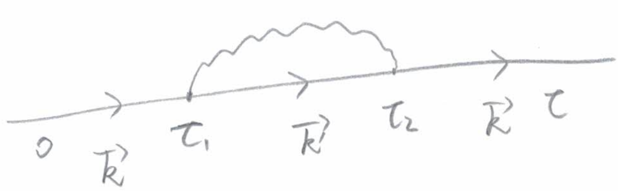
为

①


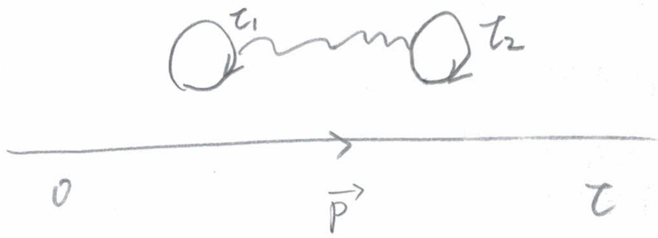
②


③


④


⑤


⑥



只计入第①和第⑤的贡献

$$G^{(2)}(\vec{p},\tau)=\Big(\frac{1}{2}\sum_{q}\int_0^{\beta}d\tau_1\int_0^{\beta}d\tau_2\, M_q M_{-q}\, G^0_{\vec{p}}(\tau,\tau_1)\, G^0_{\vec{p}-\vec{q}}(\tau_1,\tau_2)\, G^0_{\vec{p}}(\tau_2,0)$$

$$-\frac{1}{2}\sum_{q}\int_0^{\beta}d\tau_1\int_0^{\beta}d\tau_2\, M_q M_{-q}\, G^0_{\vec{p}}(\tau,\tau_2)\, G^0_{\vec{p}+\vec{q}}(\tau_2,\tau_1)\, G^0_{\vec{p}}(\tau_1,0)\Big)$$

$$\times\langle\langle(\hat{b}_{\vec{q}}(\tau_1)+\hat{b}^{+}_{-\vec{q}}(\tau_1))(\hat{b}_{-\vec{q}}(\tau_2)+\hat{b}^{+}_{\vec{q}}(\tau_2))\rangle\rangle。$$